

# On the homotopy theory of stratified spaces

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## Abstract

Let  $P$  be a poset. We show that the  $\infty$ -category  $\mathbf{Str}_P$  of  $\infty$ -categories with a conservative functor to  $P$  can be obtained from the ordinary category of  $P$ -stratified topological spaces by inverting a class of weak equivalences. For suitably nice  $P$ -stratified topological spaces, the corresponding object of  $\mathbf{Str}_P$  is the exit-path  $\infty$ -category of MacPherson, Treumann, and Lurie. In particular, the  $\infty$ -category of conically  $P$ -stratified spaces with equivalences on exit-path  $\infty$ -categories inverted embeds fully faithfully into  $\mathbf{Str}_P$ . This provides a stratified form of Grothendieck's homotopy hypothesis. We then define a combinatorial simplicial model structure on the category of simplicial sets over the nerve of  $P$  whose underlying  $\infty$ -category is the  $\infty$ -category  $\mathbf{Str}_P$ . This model structure on  $P$ -stratified simplicial sets then allows us to easily compare other theories of  $P$ -stratified spaces to ours and deduce that they all embed into ours.

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## o Introduction

The homotopy type of a topological space  $X$  is completely determined by its *underlying*  $\infty$ -groupoid with objects points of  $X$ , 1-morphisms paths in  $X$ , 2-morphisms homotopies of paths, etc. MacPherson realized that one can modify this idea to capture homotopical information about stratified topological spaces. If  $X$  is topological space with a suitably nice stratification by a poset  $P$ , then we can associate to  $X$  its *exit-path*  $\infty$ -category with objects points of  $X$ , 1-morphisms *exit paths* going from a stratum  $X_p$  to a stratum  $X_q$  for  $p \leq q$ , 2-morphisms homotopies of exit-paths respecting stratifications, etc. The adjective ‘suitably nice’ is quite important here because, while the construction of the underlying  $\infty$ -groupoid makes sense for *any* topological space, if the stratification is not sufficiently nice, then exit paths can fail to suitably compose and this informal description cannot be made to actually define an  $\infty$ -category. This is part of an overarching problem: there does not yet exist a homotopy theory of stratified spaces that is simple to define, encapsulates examples from topology, and has excellent formal properties. The purpose of this paper is to resolve this matter.

Treumann [32], Woolf [33], Lurie [HA, Appendix A], and Ayala–Francis–Rozenblyum [1, §1] have all worked to realize MacPherson’s exit-path construction under a variety of point-set topological assumptions. The takeaway from this body of work, most directly expressed by Ayala–Francis–Rozenblyum, is that the exit-path construction defines a fully faithful functor from suitably nice stratified spaces (with stratified homotopies inverted) to  $\infty$ -categories with a conservative functor to a poset, i.e.  $\infty$ -categories with a functor to a poset with fibers  $\infty$ -groupoids. We call the latter objects *abstract stratified homotopy types*. In recent work with Barwick and Glasman on stratified invariants in algebraic geometry [5], we took this as the *definition* of a homotopy theory of stratified spaces, as the ‘differential-topological’ constructions of exit-path  $\infty$ -categories are not amenable to the algebro-geometric setting, and demonstrated that abstract stratified homotopy types provide very powerful invariants of schemes.

Motivated by these bodies of work, we are led to seek a stratified version of Grothendieck’s *homotopy hypotheses*, namely, provide an *equivalence* of homotopy theories between abstract stratified homotopy types and stratified topological spaces. In various forms, this has been conjectured by Ayala–Francis–Rozenblyum [1, Conjectures o.o.4

& o.o.8], Barwick, and Woolf. The main goal of this paper is present a completely self-contained proof of a precise form of this conjecture.

## 0.1 Statement of results

The first part of our work concerns proving our ‘stratified homotopy hypothesis’. This is made possible in part by Chapter 7 of Douteau’s recent thesis [8], which realizes ideas of Henriques’ on the homotopy theory of stratified spaces [17, §4.7; 18]. Let  $\text{sd}(P)$  denote the *subdivision* of  $P$ , that is, the poset of linearly ordered finite subsets  $\Sigma \subset P$ . There is a right adjoint ‘nerve’ functor

$$\begin{aligned} N_P : \mathbf{Top}_{/P} &\rightarrow \text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet}) \\ T &\mapsto [\Sigma \mapsto \text{Sing Map}_{/P}(|\Sigma|, T)] \end{aligned}$$

from the category of  $P$ -stratified topological spaces to the category of simplicial presheaves on  $\text{sd}(P)$ . Douteau proves that the projective model structure transfers to  $\mathbf{Top}_{/P}$  along  $N_P$ , so that a morphism  $f : T \rightarrow U$  in  $\mathbf{Top}_{/P}$  is a weak equivalence (resp., fibration) if and only if for every  $\Sigma \in \text{sd}(P)$ , the induced map

$$\text{Map}_{/P}(|\Sigma|, T) \rightarrow \text{Map}_{/P}(|\Sigma|, U)$$

on topological spaces of sections over the realization of  $\Sigma$  is a weak homotopy equivalence (resp., Serre fibration).

Henriques’ insight here is that while checking equivalences after passing to exit-path  $\infty$ -categories is only reasonable for suitably nice  $P$ -stratified topological spaces, a morphism of suitably nice  $P$ -stratified topological spaces is an equivalence on exit-path  $\infty$ -categories if and only if it induces an equivalence on all spaces of sections over geometric realizations of linearly ordered finite subsets of  $P$ , and that the latter definition works well for *all* stratified topological spaces.

Even better, the resulting Quillen adjunction  $\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet}) \rightleftarrows \mathbf{Top}_{/P}$  is a simplicial Quillen equivalence of combinatorial simplicial model categories. This means that the underlying  $\infty$ -category of the Douteau–Henriques model structure on  $\mathbf{Top}_{/P}$  is equivalent to the  $\infty$ -category  $\text{Fun}(\text{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$  of presheaves of  $\infty$ -groupoids on the subdivision  $\text{sd}(P)$ . In recent joint work with Barwick and Glasman, we proved that a similarly-defined ‘nerve’ functor

$$\begin{aligned} N_P : \mathbf{Str}_P &\rightarrow \text{Fun}(\text{sd}(P)^{op}, \mathbf{Gpd}_{\infty}) \\ X &\mapsto [\Sigma \mapsto \text{Fun}_{/P}(\Sigma, X)] \end{aligned}$$

expresses the  $\infty$ -category  $\mathbf{Str}_P$  of abstract  $P$ -stratified homotopy types as an accessible localization of  $\text{Fun}(\text{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$  [5, Theorem 4.2.4]. We identify the essential image as the full subcategory of those functors  $F : \text{sd}(P)^{op} \rightarrow \mathbf{Gpd}_{\infty}$  such that the *Segal map*

$$F\{p_0 < \cdots < p_m\} \rightarrow F\{p_0 < p_1\} \times_{F\{p_1\}} F\{p_1 < p_2\} \times_{F\{p_2\}} \cdots \times_{F\{p_{m-1}\}} F\{p_{m-1} < p_m\}$$

is an equivalence for every linearly ordered finite subset  $\{p_0 < \cdots < p_m\} \subset P$ . Consequently we arrive at our ‘stratified homotopy hypothesis’:

**o.1.1 Theorem (Theorem 1.3.10).** *Let  $P$  be a poset. Then the  $\infty$ -category  $\mathbf{Str}_P$  of abstract  $P$ -stratified homotopy types is equivalent to an accessible localization of the underlying  $\infty$ -category of the combinatorial simplicial model category  $\mathbf{Top}_{/P}$  in the Douteau–Henriques model structure.*

That is to say, the  $\infty$ -category  $\mathbf{Str}_P$  of abstract  $P$ -stratified homotopy types can be obtained from the ordinary category  $\mathbf{Top}_{/P}$  of  $P$ -stratified topological spaces by inverting a class of weak equivalences (in the  $\infty$ -categorical sense).

This is a bit different than the unstratified homotopy hypothesis, as the equivalence between our homotopy theory of  $P$ -stratified topological spaces and  $\mathbf{Str}_P$  and is not just given by the formation of the exit-path  $\infty$ -category. The second part of our work reconciles this by showing that when restricted to the subcategory of  $P$ -stratified topological spaces for which Lurie’s exit-path simplicial set [HA, §A.6] is a quasicategory, the equivalence is given by the exit-path construction. We also show that previously defined homotopy theories of stratified spaces embed into  $\mathbf{Str}_P$  via the exit-path construction. In order to write down functors from these homotopy theories of stratified topological spaces into  $\mathbf{Str}_P$ , it is convenient to present  $\mathbf{Str}_P$  as the underlying  $\infty$ -category of a model category. It is not difficult to define a model structure on simplicial sets over (the nerve of)  $P$  whose fibrant objects are quasicategories with a conservative functor to  $P$ : we take the left Bousfield localization of the Joyal model structure inherited on the overcategory  $s\mathbf{Set}_{/P}$  that inverts all simplicial homotopies  $X \times \Delta^1 \rightarrow Y$  respecting the stratifications of  $X$  and  $Y$  by  $P$ . We call the resulting model structure on  $s\mathbf{Set}_{/P}$  the *Joyal–Kan model structure*. What is more surprising is that the Joyal–Kan model structure shares many of the excellent formal properties of the Kan model structure that the Joyal model structure lacks; namely, the Joyal–Kan model structure is *simplicial*. The following two theorems summarize the main features of the Joyal–Kan model structure.

**o.1.2 Theorem.** *Let  $P$  be a poset.*

- *There exists a left proper combinatorial simplicial model structure on the overcategory  $s\mathbf{Set}_{/P}$  called the Joyal–Kan model structure with cofibrations monomorphisms and fibrant objects the quasicategories  $X$  over  $P$  such that the structure morphism  $X \rightarrow P$  is a conservative functor (Propositions 2.1.4 and 2.2.4 and Theorem 2.5.10).*
- *If  $f : X \rightarrow Y$  a morphism in  $s\mathbf{Set}_{/P}$  and all of the fibers of  $X$  and  $Y$  over points of  $P$  are Kan complexes (e.g.,  $X$  and  $Y$  are fibrant objects), then  $f$  is an equivalence in the Joyal–Kan model structure if and only if  $f$  is an equivalence in the Joyal model structure (Proposition 2.5.4).*

**o.1.3 Theorem (Corollary 2.5.11).** *Let  $P$  be a poset. Then the underlying  $\infty$ -category of the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  is the  $\infty$ -category  $\mathbf{Str}_P$  of abstract  $P$ -stratified homotopy types.*

Consider the full subcategory  $\mathbf{Top}_{/P}^{\text{ex}} \subset \mathbf{Top}_{/P}$  of those  $P$ -stratified topological spaces  $T$  for which Lurie’s exit-path simplicial set  $\text{Sing}_P(T)$  is a quasicategory, hence defines a Joyal–Kan fibrant object of  $s\mathbf{Set}_{/P}$ . If  $f : T \rightarrow U$  is a morphism in  $\mathbf{Top}_{/P}^{\text{ex}}$ , then  $f$  is a Douteau–Henriques weak equivalence if and only if  $\text{Sing}_P(f)$  is a Joyal–Kan equivalence. Theorem o.1.1 implies that if we let  $W$  denote the class of morphisms in  $\mathbf{Top}_{/P}^{\text{ex}}$  that are

sent to Joyal–Kan equivalences under  $\text{Sing}_P$ , then the induced functor of  $\infty$ -categories  $\text{Sing}_P : \mathbf{Top}_{/P}^{\text{ex}}[W^{-1}] \rightarrow \mathbf{Str}_P$  is fully faithful (Comparison 3.2.2). It is not difficult to see that  $\text{Sing}_P$  is essentially surjective, which shows that the exit-path construction provides an *equivalence* of  $\infty$ -categories  $\text{Sing}_P : \mathbf{Top}_{/P}^{\text{ex}}[W^{-1}] \simeq \mathbf{Str}_P$ . This proves a precise form of [1, Conjecture 0.0.4], and a non-smooth variant of [1, Conjecture 0.0.8].

Ayala–Francis–Rozenblyum–Tanaka define and explore *conically smooth stratified spaces* [1; 2, §3]. Their homotopy theory of  $P$ -stratified spaces is the  $\infty$ -category obtained from the category  $\mathbf{Con}_P$  of conically smooth  $P$ -stratified spaces by inverting the class  $H$  of stratified homotopies. Lurie’s exit-path simplicial set defines a functor  $\text{Sing}_P : \mathbf{Con}_P \rightarrow \mathbf{sSet}_{/P}$  landing in Joyal–Kan fibrant objects and sends stratified homotopy equivalences to Joyal–Kan equivalences, hence descends to a functor

$$\text{Sing}_P : \mathbf{Con}_P[H^{-1}] \rightarrow \mathbf{Str}_P .$$

The Ayala–Francis–Rozenblyum ‘stratified homotopy hypothesis’ states that this functor is fully faithful (see Comparison 3.2.1). Hence we have a commutative triangle of fully faithful functors

$$\begin{array}{ccc} \mathbf{Con}_P[H^{-1}] & \xrightarrow{\text{Sing}_P} & \mathbf{Str}_P \\ \downarrow & \nearrow \sim & \\ \mathbf{Top}_{/P}^{\text{ex}}[W^{-1}] & \xrightarrow{\text{Sing}_P} & \end{array} ,$$

where the vertical functor is induced by the functor  $\mathbf{Con}_P \rightarrow \mathbf{Top}_{/P}^{\text{ex}}$  forgetting conically smooth structures.

One of the major benefits of the  $\infty$ -category  $\mathbf{Top}_{/P}^{\text{ex}}[W^{-1}] \simeq \mathbf{Str}_P$  over  $\mathbf{Con}_P[H^{-1}]$  is that all *conically stratified topological spaces* fit into this framework [HA, Theorem A.6.4]. *Topologically stratified spaces* in the sense of Goresky–MacPherson [14, §1.1], in particular all Whitney stratified spaces [23; 31], are conically stratified, and the  $\infty$ -category  $\mathbf{Str}_P$  captures most, if not all, examples of differential-topological interest. On the other hand, it is still unknown whether or not every Whitney stratified space admits a conically smooth structure [1, Conjecture 0.0.7].

## 0.2 Linear overview

Section 1 is dedicated the equivalence between abstract stratified homotopy types and the homotopy theory of stratified topological spaces (Theorem 0.1.1). Subsection 1.1 briefly explains the Segal space style approach to abstract stratified homotopy types from our joint work with Barwick and Glasman [5, §4]. In §1.2 we recall the basics of stratified topological spaces and how to relate them to stratified simplicial sets via Lurie’s exit-path construction. Subsection 1.3 proves Theorem 0.1.1.

Section 2 is dedicated to the Joyal–Kan model structure on  $\mathbf{sSet}_{/P}$  and the proofs of Theorems 0.1.2 and 0.1.3. In Section 3 we use Theorems 0.1.2 and 0.1.3 to give a more direct relationship between  $\mathbf{Str}_P$  and the homotopy theory of  $P$ -stratified topological spaces for which the exit-path simplicial set is a quasicategory.

Appendix A gives an account of the construction of the Douteau–Henriques model structure on  $\mathbf{Top}_{/P}$  following Chapter 7 of Douteau’s thesis [8]. We follow Douteau’s

general narrative, though our proofs tend to be rather different. We include this material here in part because Douteau’s thesis is not publicly available, and in part because we need refinements of some of Douteau’s results.

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For a long time we had a sketch of an argument proving the main theorem, and reading Sylvain Douteau’s thesis provided the key to complete it. We thank Sylvain most heartily for generously sharing his work with us and discussing the results of his thesis.

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### 0.3 Terminology & notations

**0.3.1.** We use the language and tools of higher category theory, particularly in the model of *quasicategories*, as defined by Boardman–Vogt and developed by Joyal and Lurie [HTT; HA].

- We write  $s\mathbf{Set}$  for the category of simplicial sets and  $\mathrm{Map} : s\mathbf{Set}^{op} \times s\mathbf{Set} \rightarrow s\mathbf{Set}$  for the internal-Hom in simplicial sets.
- To avoid confusion, we call weak equivalences in the Joyal model structure on  $s\mathbf{Set}$  *Joyal equivalences* and we call weak equivalences in the Kan model structure on  $s\mathbf{Set}$  *Kan equivalences*.
- We write  $s\mathbf{Set}^{Joy}$  for the model category of simplicial sets in the Joyal model structure.
- An  $\infty$ -category here will always mean *quasicategory*; we write  $\mathbf{Cat}_\infty$  for the  $\infty$ -category of  $\infty$ -categories. We write  $\mathbf{Gpd}_\infty \subset \mathbf{Cat}_\infty$  for the  $\infty$ -category of  $\infty$ -groupoids, i.e., the  $\infty$ -category of *spaces*. In order not to overload the term ‘space’, we use ‘ $\infty$ -groupoid’ to refer to homotopy types, and ‘space’ only in reference to topological spaces.
- If  $C$  is an ordinary category, we simply write  $C \in s\mathbf{Set}$  for its nerve.
- For an  $\infty$ -category  $C$ , we write  $C^\approx \subset C$  for the maximal sub- $\infty$ -groupoid contained in  $C$ .
- Let  $C$  be an  $\infty$ -category and  $W \subset \mathrm{Mor}(C)$  a class of morphisms in  $C$ . We write  $C[W^{-1}]$  for the *localization of  $C$  at  $W$* , i.e., the initial  $\infty$ -category equipped with a functor  $C \rightarrow C[W^{-1}]$  that sends morphisms in  $W$  to equivalences [7, §7.1].

- The *underlying quasicategory* of a simplicial model category  $\mathbf{A}$  is the simplicial nerve  $N_\Delta(\mathbf{A}^\circ)$  of the full subcategory  $\mathbf{A}^\circ \subset \mathbf{A}$  spanned by the fibrant–cofibrant objects (which forms a fibrant simplicial category). In the quasicategory model, this is a presentation of the localization of  $\mathbf{A}$  at its class of weak equivalences.
- For every integer  $n \geq 0$ , we write  $[n]$  for the linearly ordered poset  $\{0 < 1 < \dots < n\}$  of cardinality  $n + 1$  (whose nerve is the simplicial set  $\Delta^n$ ).
- We denote an adjunction of categories or  $\infty$ -categories by  $F : C \rightleftarrows D : G$ , where  $F$  is the left adjoint and  $G$  is the right adjoint.
- To fix a convenient category of topological spaces, we write **Top** for the category of *numerically generated* topological spaces (also called  $\Delta$ -generated or  $I$ -generated topological spaces) [9; 10; 15; 16, §3; 30], and use the term ‘topological space’ to mean ‘numerically generated topological space’. For the present work, the category of numerically generated topological spaces is preferable to the more standard category of compactly generated weakly Hausdorff topological spaces [24, Chapter 5] because any poset in the Alexandroff topology numerically generated, whereas a poset is weakly Hausdorff if and only if it is discrete.

**0.3.2 Definition.** Let  $P$  be a poset. The category of  $P$ -*stratified simplicial sets* is the overcategory  $s\mathbf{Set}_{/P}$  of simplicial sets over (the nerve of)  $P$ . Given a  $P$ -stratified simplicial set  $f : X \rightarrow P$  and point  $p \in P$ , we write  $X_p := f^{-1}(p)$  for the  $p^{\text{th}}$  *stratum* of  $X$ .

**0.3.3 Notation.** Let  $P$  be a poset. Write  $- \rtimes_P - : s\mathbf{Set}_{/P} \times s\mathbf{Set} \rightarrow s\mathbf{Set}_{/P}$  for the standard tensoring of  $s\mathbf{Set}_{/P}$  over  $s\mathbf{Set}$ , defined on objects by sending an object  $X \in s\mathbf{Set}_{/P}$  and a simplicial set  $K \in s\mathbf{Set}$  to the product  $X \rtimes_P K := X \times K$  in  $s\mathbf{Set}$  with structure morphism induced by the projection  $X \times K \rightarrow X$ . When unambiguous we write  $\rtimes$  rather than  $\rtimes_P$ , leaving the poset  $P$  implicit.

We write  $\text{Map}_{/P} : s\mathbf{Set}_{/P}^{op} \times s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}$  for the standard simplicial enrichment, whose assignment on objects is given by

$$\text{Map}_{/P}(X, Y) := s\mathbf{Set}_{/P}(X \rtimes_P \Delta^\bullet, Y),$$

and the assignment on morphisms is the obvious one.

## 1 Abstract stratified homotopy types, décollages, & stratified topological spaces

This section is dedicated to the proof of [Theorem 0.1.1](#).

### 1.1 Abstract stratified homotopy types as décollages

In work with Barwick and Glasman [5, §4], we gave a complete Segal space style description of the  $\infty$ -category of abstract  $P$ -stratified homotopy types. In this subsection we recall this work and, for completeness, include a proof of the main comparison result ([Theorem 1.1.7](#)).

**1.1.1 Definition.** Let  $P$  be a poset. The  $\infty$ -category  $\mathbf{Str}_P$  of *abstract  $P$ -stratified homotopy types* is the full subcategory of the overcategory  $\mathbf{Cat}_{\infty/P}$  spanned by those  $\infty$ -categories over  $P$  with conservative structure morphism  $C \rightarrow P$ .

Note that the mapping  $\infty$ -groupoid  $\mathrm{Map}_{\mathbf{Str}_P}(X, Y)$  coincides with the  $\infty$ -category  $\mathrm{Fun}_P(X, Y)$  of functors  $X \rightarrow Y$  over  $P$ .

**1.1.2 Recollection.** An  $\infty$ -category can be modeled as a simplicial  $\infty$ -groupoid. There is a nerve functor  $N : \mathbf{Cat}_{\infty} \rightarrow \mathrm{Fun}(\Delta^{op}, \mathbf{Gpd}_{\infty})$  defined by

$$N(C)_m := \mathrm{Fun}(\Delta^m, C)^{\simeq}.$$

The simplicial  $\infty$ -groupoid  $N(C)$  is an example of what Rezk [28] called a *complete Segal space*, i.e., a functor  $F : \Delta^{op} \rightarrow \mathbf{Gpd}_{\infty}$  satisfying the following conditions:

► *Segal condition:* For any integer  $m \geq 1$ , the natural map

$$F_m \rightarrow F\{0 \leq 1\} \times_{F\{1\}} F\{1 \leq 2\} \times_{F\{2\}} \cdots \times_{F\{m-1\}} F\{m-1 \leq m\}$$

is an equivalence.

► *Completeness condition:* The natural morphism  $F_0 \rightarrow F_3 \times_{F\{0,2\} \times F\{1,3\}} F_0$  is an equivalence in  $\mathbf{Gpd}_{\infty}$ .

Joyal and Tierney showed that the nerve is fully faithful with essential image the full subcategory  $\mathbf{CSS}$  of complete Segal spaces spanned by the complete Segal spaces [21]. We can isolate the  $\infty$ -groupoids in  $\mathbf{CSS}$ : an  $\infty$ -category  $C$  is an  $\infty$ -groupoid if and only if  $N(C) : \Delta^{op} \rightarrow \mathbf{Gpd}_{\infty}$  is left Kan extended from  $\{0\} \subset \Delta^{op}$ .

We now give an analogous description of  $\mathbf{Str}_P$ .

**1.1.3 Notation.** Let  $P$  be a poset. We write  $\mathrm{sd}(P)$  for the *subdivision* of  $P$  – that is,  $\mathrm{sd}(P)$  is the poset of nonempty linearly ordered finite subsets  $\Sigma \subset P$  ordered by containment. We call a nonempty linearly ordered finite subsets  $\Sigma \subset P$  of  $P$  a *string* in  $P$ .

**1.1.4 Definition.** Let  $P$  be a poset. A functor  $F : \mathrm{sd}(P)^{op} \rightarrow \mathbf{Gpd}_{\infty}$  is a *décollage* (over  $P$ ) if and only if, for every string  $\{p_0 < \cdots < p_m\} \subset P$ , the map

$$F\{p_0 < \cdots < p_m\} \rightarrow F\{p_0 < p_1\} \times_{F\{p_1\}} F\{p_1 < p_2\} \times_{F\{p_2\}} \cdots \times_{F\{p_{m-1}\}} F\{p_{m-1} < p_m\}$$

is an equivalence. We write

$$\mathbf{Déc}_P \subset \mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$$

for the full subcategory spanned by the décollages. Note that  $\mathbf{Déc}_P$  is closed under limits and filtered colimits in  $\mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$ .

A nerve style construction provides an equivalence  $\mathbf{Str}_P \simeq \mathbf{Déc}_P$ .

**1.1.5 Construction.** Let  $P$  be a poset. We have a fully faithful functor  $\mathrm{sd}(P) \hookrightarrow \mathbf{Str}_P$  given by regarding a string  $\Sigma$  as an  $\infty$ -category over  $P$  via the inclusion  $\Sigma \hookrightarrow P$ . Define a functor  $N_P : \mathbf{Str}_P \rightarrow \mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$  by the assignment

$$X \mapsto [\Sigma \mapsto \mathrm{Map}_{\mathbf{Str}_P}(\Sigma, X)].$$



**1.1.6.** Since a string  $\{p_0 < \dots < p_n\}$  is the iterated pushout

$$\{p_0 < p_1\} \cup^{\{p_1\}} \dots \cup^{\{p_{n-1}\}} \{p_{n-1} < p_n\}$$

in  $\mathbf{Str}_P$ , the functor  $N_P$  lands in the full subcategory  $\mathbf{D\acute{e}c}_P$ .

**1.1.7 Theorem** ([5, Theorem 4.2.4]). *For any poset  $P$ , the functor  $N_P : \mathbf{Str}_P \rightarrow \mathbf{D\acute{e}c}_P$  is an equivalence of  $\infty$ -categories.*

*Proof.* Let  $\Delta_P$  denote the category of simplices of  $P$ . The Joyal–Tierney theorem [21] implies that the nerve functor

$$\begin{aligned} \mathbf{Cat}_{\infty/P} &\rightarrow \mathbf{Fun}(\Delta_P^{op}, \mathbf{Gpd}_{\infty})_{/NP} \simeq \mathbf{Fun}(\Delta_P^{op}, \mathbf{Gpd}_{\infty}) \\ X &\mapsto [\Sigma \mapsto \mathbf{Fun}_{/P}(\Sigma, X)^{\simeq}] \end{aligned}$$

is fully faithful, with essential image  $\mathbf{CSS}_{/NP}$  those functors  $\Delta_P^{op} \rightarrow \mathbf{Gpd}_{\infty}$  that satisfy both the Segal condition and the completeness condition. Now notice that left Kan extension along the inclusion  $\mathbf{sd}(P) \hookrightarrow \Delta_P$  defines a fully faithful functor  $\mathbf{D\acute{e}c}_P \hookrightarrow \mathbf{CSS}_{/NP}$  whose essential image consists of those complete Segal spaces  $C \rightarrow NP$  such that for any  $p \in P$ , the complete Segal space  $C_p$  is an  $\infty$ -groupoid.  $\square$

**1.1.8.** Since  $\mathbf{Str}_P$  is presentable and  $\mathbf{D\acute{e}c}_P \subset \mathbf{Fun}(\mathbf{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$  is closed under limits and filtered colimits, the Adjoint Functor Theorem shows that the nerve expresses the  $\infty$ -category  $\mathbf{Str}_P$  as an  $\omega$ -accessible localization of  $\mathbf{Fun}(\mathbf{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$ .

**1.1.9.** The equivalence [Theorem 1.1.7](#) implies that equivalences in  $\mathbf{Str}_P$  are checked on strata and *links*. That is, a morphism  $f : X \rightarrow Y$  in  $\mathbf{Str}_P$  is an equivalence if and only if  $f$  induces an equivalence on strata and for each pair  $p < q$  in  $P$ , the induced map on *links*

$$\mathbf{Map}_{\mathbf{Str}_P}(\{p < q\}, X) \rightarrow \mathbf{Map}_{\mathbf{Str}_P}(\{p < q\}, Y)$$

is an equivalence in  $\mathbf{Gpd}_{\infty}$ . (This can also easily be proven directly without appealing to [Theorem 1.1.7](#).)

## 1.2 Recollections on stratified topological spaces

We now recall the relationship between  $P$ -stratified topological spaces and  $P$ -stratified simplicial sets. Please recall that we write  $\mathbf{Top}$  for the category of *numerically generated* topological spaces [\(0.3.1\)](#).

**1.2.1 Recollection.** The *Alexandroff topology* on a poset  $P$  is the topology on the underlying set of  $P$  in which a subset  $U \subset P$  is open if and only if  $x \in U$  and  $y \geq x$  implies that  $y \in U$ .

**1.2.2.** Please note that every poset in the Alexandroff topology is a numerically generated topological space.

**1.2.3 Definition.** Let  $P$  be a poset. We simply write  $P \in \mathbf{Top}$  for the set  $P$  equipped with the Alexandroff topology. The category of  *$P$ -stratified topological spaces* is the overcategory  $\mathbf{Top}_{/P}$ . If  $s : T \rightarrow P$  is a  $P$ -stratified topological space, for each  $p \in P$  we write  $T_p := s^{-1}(p)$  for the  $p^{\text{th}}$  *stratum* of  $T$ .

**1.2.4 Notation.** Let  $B$  be a topological space, and  $T, U \in \mathbf{Top}_{/B}$ . We write  $\mathrm{Map}_{/B}(T, U)$  for the topological space of maps  $T \rightarrow U$  over  $B$ . If we need to clarify notation, we write  $\mathrm{Map}_{\mathbf{Top}_{/B}}(T, U)$  for this topological space.

For any topological space  $V$ , we write  $T \rtimes_B V$  or simply  $T \rtimes V$  for the object of  $\mathbf{Top}_{/B}$  given by the product  $T \times V$  with structure morphism induced by the projection  $T \times V \rightarrow T$ .

**1.2.5.** Let  $P$  be a poset. Then since the subdivision  $\mathrm{sd}(P)$  of  $P$  is the category of nondegenerate simplices of  $P$ , the poset  $P$  is the colimit  $\mathrm{colim}_{\Sigma \in \mathrm{sd}(P)} \Sigma$  in the category of posets (equivalently, in  $\mathbf{sSet}$ ).

**1.2.6 Recollection** ([HA, §A.6]). Let  $P$  be a poset. There is a natural stratification

$$\pi_P : |P| \rightarrow P$$

of the geometric realization of (the nerve of)  $P$  by the Alexandroff space  $P$ . This is defined by appealing to (1.2.5), which implies that it suffices to give the standard topological  $n$ -simplex  $|\Delta^n|$  a  $[n]$ -stratification natural in  $[n]$ ; this is given by the map  $|\Delta^n| \rightarrow [n]$  defined by

$$(t_0, \dots, t_n) \mapsto \max \{ i \in [n] \mid t_i \neq 0 \} .$$

If  $X$  is a  $P$ -stratified simplicial set, then we can stratify the geometric realization  $|X|$  by composing the structure morphism  $|X| \rightarrow |P|$  with  $\pi_P$ . This defines a left adjoint functor  $|-|_P : \mathbf{sSet}_{/P} \rightarrow \mathbf{Top}_{/P}$  with right adjoint  $\mathrm{Sing}_P : \mathbf{Top}_{/P} \rightarrow \mathbf{sSet}_{/P}$  computed by the pullback of simplicial sets

$$\mathrm{Sing}_P(T) := P \times_{\mathrm{Sing}(P)} \mathrm{Sing}(T) ,$$

where the morphism  $P \rightarrow \mathrm{Sing}(P)$  is adjoint to  $\pi_P$ .

**1.2.7.** Let  $T$  be a  $P$ -stratified topological space. Then for each  $p \in P$ , the stratum  $\mathrm{Sing}_P(T)_p$  is isomorphic to the Kan complex  $\mathrm{Sing}(T_p)$ .

**1.2.8.** Lurie proves [HA, Theorem A.6.4] that if  $T \in \mathbf{Top}_{/P}$  is *conically stratified*<sup>1</sup>, then the simplicial set  $\mathrm{Sing}_P(T)$  is a quasicategory.

We will use the following observation repeatedly throughout this text.

**1.2.9 Remark.** Let  $P$  be a poset. Then the adjunction  $|-|_P : \mathbf{sSet}_{/P} \rightleftarrows \mathbf{Top}_{/P} : \mathrm{Sing}_P$  is simplicial. That is, if  $X$  is a  $P$ -stratified simplicial set and  $T$  be a  $P$ -stratified topological space, then we have a natural isomorphism of simplicial sets

$$\mathrm{Sing}(\mathrm{Map}_{\mathbf{Top}_{/P}}(|X|_P, T)) \cong \mathrm{Map}_{\mathbf{sSet}_{/P}}(X, \mathrm{Sing}_P(T)) .$$

---

<sup>1</sup>See [HA, Definitions A.5.3 & A.5.5] for the definition of a conically stratified topological space.

### 1.3 Stratified topological spaces as décollages

In this subsection we prove [Theorem 0.1.1](#). First we set some notation for the adjunction relating  $P$ -stratified topological spaces and simplicial presheaves on the subdivision of  $P$  and recall Douteau's Transfer Theorem ([Theorem 1.3.5](#)).

**1.3.1 Notation.** Let  $P$  be a poset. We write  $N_P : \mathbf{sSet}_{/P} \rightarrow \text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})$  for the functor given by the assignment

$$X \mapsto [\Sigma \mapsto \text{Map}_{/P}(\Sigma, X)]$$

The functor  $N_P$  admits a left adjoint  $L_P : \text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet}) \rightarrow \mathbf{sSet}_{/P}$  given by the left Kan extension of the Yoneda embedding  $\text{sd}(P) \hookrightarrow \text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})$  along the fully faithful functor  $\text{sd}(P) \hookrightarrow \mathbf{sSet}_{/P}$  given by  $\Sigma \mapsto [\Sigma \subset P]$ . Thus  $L_P$  is given by the coend formula<sup>2</sup>

$$L_P(F) \cong \int^{\Sigma \in \text{sd}(P)} \Sigma \rtimes F(\Sigma) .$$

**1.3.2.** Let  $P$  be a poset. Write  $\text{Pair}(P) \subset \text{sd}(P)^{op} \times \text{sd}(P)$  for the full subposet spanned by those pairs  $(\Sigma, \Sigma')$  where  $\Sigma' \subset \Sigma$ . The poset  $\text{Pair}(P)$  is an explicit description of the opposite of the twisted arrow category of  $\text{sd}(P)^{op}$ . Hence by the formula for a coend in terms of a colimit over twisted arrow categories (see [22, Chapter XI, §5, Proposition 1]), the value of the left adjoint  $L_P$  on a functor  $F : \text{sd}(P)^{op} \rightarrow \mathbf{sSet}$  is given by the colimit

$$L_P(F) \cong \text{colim}_{(\Sigma, \Sigma') \in \text{Pair}(P)} \Sigma' \rtimes F(\Sigma) .$$

This more concrete description of  $L_P$  will be of great utility in [Appendix A](#).

**1.3.3 Notation.** Write  $D_P : \mathbf{Top}_{/P} \rightarrow \text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})$  for the right adjoint functor given by the composite  $N_P \circ \text{Sing}_P$ . It follows from [Remark 1.2.9](#) that  $D_P$  is given by the assignment

$$T \mapsto [\Sigma \mapsto \text{Sing Map}_{\mathbf{Top}_{/P}}(|\Sigma|_P, T)] .$$

**1.3.4 Notation.** Let  $P$  be a poset. We write  $\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})^{proj}$  the functor category  $\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})$  equipped with the projective model structure with respect to the Kan model structure on  $\mathbf{sSet}$ .

We are now ready to state Douteau's Transfer Theorem, and use it to prove that the  $\infty$ -category  $\mathbf{Str}_P$  of abstract  $P$ -stratified homotopy types is an  $\omega$ -accessible localization of the underlying  $\infty$ -category of  $\mathbf{Top}_{/P}$ .

**1.3.5 Theorem** (Douteau [8, Théorèmes 7.2.1, 7.3.7, 7.3.8 & 7.3.10]; [Corollary A.2.9](#) and [Theorem A.4.10](#)). *For any poset  $P$ , the projective model structure on  $\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})$  right-transfers<sup>3</sup> to  $\mathbf{Top}_{/P}$  along the simplicial adjunction*

$$(1.3.6) \quad |-|_P \circ L_P : \text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})^{proj} \rightleftarrows \mathbf{Top}_{/P} : D_P .$$

*Moreover, with respect to these model structures, the adjunction (1.3.6) is a simplicial Quillen equivalence of combinatorial simplicial model categories.*

<sup>2</sup>See [29, §1.5].

<sup>3</sup>We review right-transferred model structures in [Appendix A](#).

We refer to this model structure as the **Douteau–Henriques** model structure.

**1.3.7.** Explicitly, the Douateau–Henriques model structure on  $\mathbf{Top}_{/P}$  admits the following description:

- (1.3.7.1) A morphism  $f : T \rightarrow U$  in  $\mathbf{Top}_{/P}$  is a Douateau–Henriques fibration if and only if for every string  $\Sigma \subset P$ , the induced map of topological spaces

$$\mathrm{Map}_{/P}(|\Sigma|_P, T) \rightarrow \mathrm{Map}_{/P}(|\Sigma|_P, U)$$

is a Serre fibration.

- (1.3.7.2) A morphism  $f : T \rightarrow U$  in  $\mathbf{Top}_{/P}$  is a Douateau–Henriques weak equivalence if and only if for every string  $\Sigma \subset P$ , the induced map of topological spaces

$$\mathrm{Map}_{/P}(|\Sigma|_P, T) \rightarrow \mathrm{Map}_{/P}(|\Sigma|_P, U)$$

is a weak homotopy equivalence of topological spaces.

- (1.3.7.3) The sets

$$\{ |\Sigma \rtimes \partial \Delta^n|_P \hookrightarrow |\Sigma \rtimes \Delta^n|_P \mid \Sigma \in \mathrm{sd}(P), n \geq 0 \}$$

and

$$\{ |\Sigma \rtimes \Lambda_k^n|_P \hookrightarrow |\Sigma \rtimes \Delta^n|_P \mid \Sigma \in \mathrm{sd}(P), n \geq 0 \text{ and } 0 \leq k \leq n \}$$

are generating sets of Douateau–Henriques cofibrations and trivial cofibrations, respectively.

**1.3.8.** The  $\infty$ -category  $\mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$  of presheaves of  $\infty$ -groupoids on the subdivision  $\mathrm{sd}(P)^{op}$  is the underlying  $\infty$ -category of the combinatorial simplicial model category  $\mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{sSet})^{proj}$  [HTT, Proposition 4.2.4.4]. Hence the simplicial Quillen equivalence (1.3.6) provides an equivalence of  $\infty$ -categories between the underlying  $\infty$ -category of  $\mathbf{Top}_{/P}$  and  $\mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$ .

**1.3.9.** **Theorem 1.1.7** and **Remark 1.2.9** show that if  $f : T \rightarrow U$  is a morphism in  $\mathbf{Top}_{/P}$  and both  $\mathrm{Sing}_P(T)$  and  $\mathrm{Sing}_P(U)$  are quasicategories, then  $f$  is Douateau–Henriques equivalence if and only if  $\mathrm{Sing}_P(f)$  is an equivalence when regarded as a morphism in the  $\infty$ -category  $\mathbf{Str}_P$  of abstract  $P$ -stratified homotopy types.

We now arrive at our ‘stratified homotopy hypothesis’:

**1.3.10 Theorem.** *Let  $P$  be a poset. Then the  $\infty$ -category  $\mathbf{Str}_P$  is equivalent to an  $\omega$ -accessible localization of the underlying  $\infty$ -category of the combinatorial simplicial model category  $\mathbf{Top}_{/P}$ .*

*Proof.* Since the underlying  $\infty$ -category of  $\mathbf{Top}_{/P}$  is equivalent to  $\mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$  and  $\mathbf{Str}_P$  is an  $\omega$ -accessible localization of  $\mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{Gpd}_{\infty})$  (1.1.8), we deduce that  $\mathbf{Str}_P$  is an  $\omega$ -accessible localization of the underlying  $\infty$ -category of  $\mathbf{Top}_{/P}$ .  $\square$

**1.3.11.** Since the model category  $\text{Fun}(\text{sd}(P)^{op}, s\mathbf{Set})^{proj}$  is *left proper*, there exists a left Bousfield localization of the projective model structure presenting  $\mathbf{D}\acute{e}c_P \simeq \mathbf{Str}_P$  [HTT, Proposition A.3.7.8]. We do not, however, know whether or not the Douteau–Henriques model structure on  $\mathbf{Top}_P$  is left proper. So while there does exist a left Bousfield localization of  $\mathbf{Top}_P$  presenting the  $\infty$ -category  $\mathbf{Str}_P$ , we only know that it exists as a *left model category*, and it may not exist as a model category [3, §5].

Either way, **Theorem 1.3.10** shows that  $\infty$ -category can be obtained from the ordinary category  $\mathbf{Top}_P$  of  $P$ -stratified topological spaces by inverting a class of weak equivalences (in the  $\infty$ -categorical sense).

## 2 The Joyal–Kan model structure

In this section we define a combinatorial simplicial model structure on  $s\mathbf{Set}_P$  that presents the  $\infty$ -category  $\mathbf{Str}_P$ . **Subsections 2.1 to 2.3** explore the basic properties of this model structure, and **Subsections 2.4 and 2.5** are dedicated to proving it is simplicial.

### 2.1 Definition

In this subsection we define a *Joyal–Kan* model structure on simplicial sets stratified over a poset  $P$  by taking the left Bousfield localization of the Joyal model structure that inverts those simplicial homotopies  $X \times \Delta^1 \rightarrow Y$  over  $P$  respecting stratifications.

**2.1.1 Notation.** Let  $P$  be a poset. Write  $E_P$  for the set of morphisms in  $s\mathbf{Set}_P$  consisting of the endpoint inclusions  $\Delta^{[0]}, \Delta^{[1]} \subset \Delta^1$  over  $P$  for which the stratification  $f: \Delta^1 \rightarrow P$  is constant.

**2.1.2 Definition.** Let  $P$  be a poset. The *Joyal–Kan model structure* on  $s\mathbf{Set}_P$  is the  $s\mathbf{Set}^{Joy}$ -enriched left Bousfield localization of the Joyal model structure on  $s\mathbf{Set}_P$  with respect to the set  $E_P$ .

We now proceed to verify that the Joyal–Kan model structure exists as well as explore its basic properties.

**2.1.3 Remark.** Let  $P$  be a poset,  $i: X \rightarrow Y$  a morphism in  $s\mathbf{Set}_P$ , and  $j: A \rightarrow B$  a morphism of simplicial sets. Then on underlying simplicial sets, the pushout-tensor

$$i \rtimes j: (X \rtimes B) \sqcup^{X \rtimes A} (Y \rtimes A) \rightarrow Y \rtimes B$$

is simply the pushout-product

$$i \hat{\rtimes} j: (X \times B) \sqcup^{X \times A} (Y \times A) \rightarrow Y \times B$$

in  $s\mathbf{Set}$ . Since the pushout-product of monomorphisms in  $s\mathbf{Set}$  is a monomorphism and the forgetful functor  $s\mathbf{Set}_P \rightarrow s\mathbf{Set}$  detects monomorphisms, if  $i$  and  $j$  are monomorphisms, then  $i \hat{\rtimes} j$  is a monomorphism.

Since the Joyal model structure on  $s\mathbf{Set}_P$  is  $s\mathbf{Set}^{Joy}$ -enriched, a direct application of [3, Theorems 4.7 & 4.46] shows that the Joyal–Kan model structure on  $s\mathbf{Set}_P$  exists and satisfies the expected properties which we summarize in **Proposition 2.1.4**.

**2.1.4 Proposition.** *Let  $P$  be a poset. The Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  exists and satisfies the following properties.*

- (2.1.4.1) *The Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  is combinatorial.*
- (2.1.4.2) *The Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  is  $s\mathbf{Set}^{\text{Joy}}$ -enriched.*
- (2.1.4.3) *The cofibrations of in the Joyal–Kan model structure are precisely the monomorphisms of simplicial sets; in particular, the Joyal–Kan model structure is left proper.*
- (2.1.4.4) *The fibrant objects in the Joyal–Kan model structure are precisely the fibrant objects in the Joyal model structure on  $s\mathbf{Set}_{/P}$  that are also  $E_P$ -local.*
- (2.1.4.5) *The weak equivalences in the Joyal–Kan model structure are the  $E_P$ -local weak equivalences.*

**2.1.5 Remark.** When  $P = *$  is the terminal poset, the Joyal–Kan model structure on  $s\mathbf{Set} = s\mathbf{Set}_{/*}$  coincides with the Kan model structure.

## 2.2 Fibrant objects in the Joyal–Kan model structure

We now identify the fibrant objects in the Joyal–Kan model structure.

**2.2.1 Recollection.** By [HTT, Corollary 2.4.6.5] if  $C$  is a quasicategory, then a morphism of simplicial sets  $f: X \rightarrow C$  is a fibration in the Joyal model structure on  $s\mathbf{Set}$  if and only if the following conditions are satisfied:

- (2.2.1.1) The morphism  $f$  is an inner fibration.
- (2.2.1.2) For every equivalence  $e: c \simeq c'$  in  $C$  and object  $\tilde{c} \in X$  such that  $f(\tilde{c}) = c$ , there exists an equivalence  $\tilde{e}: \tilde{c} \simeq \tilde{c}'$  in  $X$  such that  $f(\tilde{e}) = e$ .

A morphism of simplicial sets satisfying (2.2.1.1) and (2.2.1.2) is called an **isofibration**. (See also [6, §2].)

We make use of the following obvious fact.

**2.2.2 Lemma.** *Let  $C$  be a quasicategory whose equivalences are precisely the degenerate edges (e.g., a poset). Then a morphism of simplicial sets  $f: X \rightarrow C$  is an isofibration if and only if  $f$  is an inner fibration.*

**2.2.3 Proposition.** *Let  $P$  be a poset. An object  $X$  of  $s\mathbf{Set}_{/P}$  is fibrant in the Joyal–Kan model structure if and only if the structure morphism  $X \rightarrow P$  is an inner fibration and for every  $p \in P$  the stratum  $X_p$  is a Kan complex.*

*Proof.* Since the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  is the left Bousfield localization of the Joyal model structure on  $s\mathbf{Set}_{/P}$  with respect to  $E_P$ , the fibrant objects in the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  are the fibrant objects in the Joyal model structure on  $s\mathbf{Set}_{/P}$  that are also  $E_P$ -local. An object  $X \in s\mathbf{Set}_{/P}$  is fibrant in the Joyal model structure

if and only if the structure morphism  $X \rightarrow P$  is an isofibration, or, equivalently the structure morphism  $X \rightarrow P$  is an inner fibration ([Lemma 2.2.2](#)).

Now we analyze the  $E_P$ -locality condition. A Joyal-fibrant object  $X \in \mathbf{sSet}_{/P}$  is  $E_P$ -local if and only if for every 1-simplex  $\sigma: \Delta^1 \rightarrow P$  such that  $\sigma(0) = \sigma(1)$ , evaluation morphisms

$$\mathrm{ev}_i: \mathrm{Map}_{/P}(\Delta^1, X) \rightarrow \mathrm{Map}_{/P}(\Delta^{[i]}, X)$$

for  $i = 0, 1$  are isomorphisms in the homotopy category of  $\mathbf{sSet}^{\mathrm{Joyal}}$ . Let  $p \in P$  be such that  $\sigma(0) = \sigma(1) = p$ . Then

$$\mathrm{Map}_{/P}(\Delta^1, X) \cong \mathrm{Map}(\Delta^1, X_p)$$

and

$$\mathrm{Map}_{/P}(\Delta^{[i]}, X) \cong \mathrm{Map}(\Delta^{[i]}, X_p),$$

for  $i = 0, 1$ . Under these identifications, the evaluation morphisms

$$\mathrm{ev}_i: \mathrm{Map}_{/P}(\Delta^1, X) \rightarrow \mathrm{Map}_{/P}(\Delta^{[i]}, X)$$

are identified with the evaluation morphisms

$$\mathrm{ev}_i: \mathrm{Map}(\Delta^1, X_p) \rightarrow \mathrm{Map}(\Delta^{[i]}, X_p) \cong X_p,$$

for  $i = 0, 1$ . Since the strata of  $X$  are quasicategories,  $X$  is  $E_P$ -local if and only if for every  $p \in P$ , the evaluation morphisms

$$\mathrm{ev}_i: \mathrm{Map}(\Delta^1, X_p) \rightarrow \mathrm{Map}(\Delta^{[i]}, X_p) \cong X_p,$$

for  $i = 0, 1$ , are Joyal equivalences. To conclude, recall that for a quasicategory  $C$ , the evaluation morphisms  $\mathrm{ev}_0, \mathrm{ev}_1: \mathrm{Map}(\Delta^1, C) \rightarrow C$  are Joyal equivalences if and only if  $C$  is a Kan complex.  $\square$

Combining [Proposition 2.2.3](#) with [[HTT](#), Proposition 2.3.1.5] we deduce:

**2.2.4 Proposition.** *Let  $P$  be a poset,  $X$  a simplicial set, and  $f: X \rightarrow P$  a morphism of simplicial sets. The following are equivalent:*

- (2.2.4.1) *The object  $f: X \rightarrow P$  of  $\mathbf{sSet}_{/P}$  is fibrant in the Joyal–Kan model structure.*
- (2.2.4.2) *The morphism  $f: X \rightarrow P$  is an inner fibration with all fibers Kan complexes.*
- (2.2.4.3) *The simplicial set  $X$  is a quasicategory and all of the fibers of  $f$  are Kan complexes.*
- (2.2.4.4) *The simplicial set  $X$  is a quasicategory and  $f$  is a conservative functor between quasicategories.*

A number of facts are now immediate.

**2.2.5 Corollary.** *Let  $P$  be a poset. The set of equivalences between two fibrant objects in the Joyal–Kan model structure on  $\mathbf{sSet}_{/P}$  is the set of Joyal equivalences over  $P$ , i.e., fully faithful and essentially surjective functors over  $P$ .*

**2.2.6 Corollary.** *Let  $P$  be a poset. A morphism in  $\mathbf{sSet}_{/P}$  between fibrant objects in the Joyal–Kan model structure is conservative functor.*

*Proof.* Note that if a composite functor  $gf$  is conservative and  $g$  is conservative, then  $f$  is conservative.  $\square$

## 2.3 Stratified horn inclusions

In this subsection we characterize the horn inclusions in  $\mathbf{sSet}_{/P}$  that are Joyal–Kan equivalences. We will use these horn inclusions in our proof that the Joyal–Kan model structure is simplicial (see §2.4).

**2.3.1 Proposition.** *Let  $P$  be a poset. A horn inclusion  $i: \Lambda_k^n \hookrightarrow \Delta^n$  over  $P$  stratified by a morphism  $f: \Delta^n \rightarrow P$  is a Joyal–Kan equivalence in  $\mathbf{sSet}_{/P}$  if and only if one of the following conditions holds:*

$$(2.3.1.1) \quad 0 < k < n.$$

$$(2.3.1.2) \quad k = 0 \text{ and } f(0) = f(1).$$

$$(2.3.1.3) \quad k = n \text{ and } f(n-1) = f(n).$$

*Proof.* First we show that the class horn inclusions (2.3.1.1)–(2.3.1.3) are Joyal–Kan equivalences. It is clear that inner horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  are weak equivalences in the Joyal–Kan model structure on  $\mathbf{sSet}_{/P}$  as they are already weak equivalences in the Joyal model structure. If  $n = 1$ , then the endpoint inclusions  $\Lambda_0^1, \Lambda_1^1 \hookrightarrow \Delta^1$  where  $f(0) = f(1)$  are Joyal–Kan equivalences by the definition of the Joyal–Kan model structure.

Now we tackle the case of higher outer horns. We treat the case of left horns  $\Lambda_0^n \hookrightarrow \Delta^n$  where the stratification  $f: \Delta^n \rightarrow P$  has the property that  $f(0) = f(1)$  (i.e., the class specified by (2.3.1.2)); the case of right horns is dual. We prove the claim by induction on  $n$ .

For the base case where  $n = 2$ , write  $D_0^2$  for the (nerve of the) preorder given by  $0 \leq 1 \leq 2$  along with  $0 \geq 1$ , and stratify  $D_0^2$  by the unique extension of  $f$  to  $D_0^2$ . All stratifications will be induced by  $f$  via inclusions into  $D_0^2$ . We prove the claim by showing that the inclusions  $\Lambda_0^2, \Delta^2 \hookrightarrow D_0^2$  are Joyal–Kan equivalences and conclude by the 2-of-3 property. Write  $E$  for the walking isomorphism category  $0 \cong 1$  and consider the cube

$$(2.3.2) \quad \begin{array}{ccccc} & & \Delta^{\{0\}} & \xrightarrow{\quad} & \Delta^{\{0,2\}} \\ & \nearrow & \downarrow \wr & & \downarrow \wr \\ \Delta^{\{0\}} & \xrightarrow{\quad} & \Delta^{\{0,2\}} & & \\ \downarrow \wr & & \downarrow \wr & \searrow \wr & \\ & \Delta^{\{0,1\}} & \xrightarrow{\quad} & \Lambda_0^2 & \\ \downarrow \wr & \nearrow & \downarrow \wr & \nearrow & \\ E & \xrightarrow{\quad} & L_0^2 & & \end{array},$$

where the front face is a pushout defining the simplicial set  $L_0^2$  and the back face is a pushout square. Since  $f(0) = f(1)$ , the inclusion  $\Delta^{\{0\}} \hookrightarrow \Delta^{\{0,1\}}$  is a trivial Joyal–Kan cofibration; the fact that the back face of (2.3.2) is a pushout shows that the inclusion  $\Delta^{\{0,2\}} \hookrightarrow \Lambda_0^2$  is a trivial Joyal–Kan cofibration. Since the inclusion  $\Delta^{\{0\}} \hookrightarrow E$  is a trivial Joyal cofibration and the front face of (2.3.2) is a pushout, the inclusion  $\Delta^{\{0,2\}} \hookrightarrow L_0^2$  is a



trivial Joyal cofibration. By the 2-of-3 property, the induced map on pushouts  $\Lambda_0^2 \hookrightarrow L_0^2$  is a trivial Joyal–Kan cofibration. Similarly, the inclusion

$$\Lambda_1^2 \hookrightarrow L_1^2 := E \cup^{\Delta^{(1)}} \Delta^{\{1,2\}}$$

is a trivial Joyal–Kan cofibration. The inclusions  $L_0^2, L_1^2 \hookrightarrow D_0^2$  are inner anodyne, so in particular the composite inclusion

$$\Lambda_0^2 \hookrightarrow L_0^2 \hookrightarrow D_0^2$$

is a trivial Joyal–Kan cofibration. Finally, to see that the inclusion  $\Delta^2 \hookrightarrow D_0^2$  is a Joyal–Kan equivalence note that we have a commutative square

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{\sim} & \Delta^2 \\ \downarrow & & \downarrow \\ L_1^2 & \xrightarrow{\sim} & D_0^2, \end{array}$$

where the horizontal morphisms are inner anodyne and the inclusion  $\Lambda_1^2 \hookrightarrow L_1^2$  is a Joyal–Kan equivalence. This concludes the base case.

Now we prove the induction step with  $n \geq 3$  and  $\Lambda_0^n \hookrightarrow \Delta^n$  an outer horn inclusion over  $P$  where the stratification  $f: \Delta^n \rightarrow P$  has the property that  $f(0) = f(1)$ . Write

$$\Lambda_{\{0,2\}}^n := \bigcup_{j \in [n] \setminus \{0,2\}} \Delta^{\{0, \dots, j-1, j+1, \dots, n\}} \subset \Delta^n$$

and note that by [4, Lemma 12.1.3] the inclusion  $\Lambda_{\{0,2\}}^n \hookrightarrow \Delta^n$  is inner anodyne. Since we have a factorization of the inclusion  $\Lambda_{\{0,2\}}^n \hookrightarrow \Delta^n$  as a composite

$$\Lambda_{\{0,2\}}^n \hookrightarrow \Lambda_0^n \hookrightarrow \Delta^n,$$

the claim is equivalent to showing that the inclusion  $\Lambda_{\{0,2\}}^n \hookrightarrow \Lambda_0^n$  is a Joyal–Kan equivalence in  $\mathbf{sSet}_{/P}$ . To see this, note that we have a pushout square in  $\mathbf{sSet}_{/P}$

$$(2.3.3) \quad \begin{array}{ccc} \Lambda_0^{\{0,1,3,\dots,n\}} & \xrightarrow{\sim} & \Delta^{\{0,1,3,\dots,n\}} \\ \downarrow & & \downarrow \\ \Lambda_{\{0,2\}}^n & \xrightarrow{\sim} & \Lambda_0^n, \end{array}$$

where the inclusion  $\Lambda_0^{\{0,1,3,\dots,n\}} \hookrightarrow \Delta^{\{0,1,3,\dots,n\}}$  is a trivial Joyal–Kan cofibration by the induction hypothesis.

Now we prove the horn inclusions given by the classes (2.3.1.1)–(2.3.1.3) are the only horn inclusions over  $P$  that are trivial Joyal–Kan cofibrations. Equivalently, if  $i: \Lambda_k^n \hookrightarrow \Delta^n$  is an outer horn and either  $k = 0$  and  $f(0) \neq f(1)$ , or  $k = n$  and  $f(n-1) \neq f(n)$ , then  $i$  is not a Joyal–Kan equivalence. We treat the case that  $k = 0$ ; the case that  $k = n$  is dual. The cases where  $n = 1$  and  $n = 2$  require slightly different (but easier) arguments than when  $n \geq 3$ , so we tackle those first.

When  $n = 1$ , we need to show that the endpoint inclusion  $\Delta^{\{0\}} \hookrightarrow \Delta^1$  is not a Joyal–Kan fibration, where the stratification  $f: \Delta^1 \rightarrow P$  is a monomorphism. In this case, by [Proposition 2.2.4](#) both  $\Delta^{\{0\}}$  and  $\Delta^1$  are fibrant in the Joyal–Kan model structure, so by [Corollary 2.2.5](#) we just need to check that the inclusion  $i: \Delta^{\{0\}} \hookrightarrow \Delta^1$  is not a Joyal equivalence, which is clear.

For  $n = 2$ , note that the simplicial set  $\Lambda_0^2$  is a 1-category. Since  $f(0) \neq f(1)$ , we have  $f(0) \neq f(2)$ , so the functor  $f: \Lambda_0^2 \rightarrow P$  is conservative; applying [Proposition 2.2.4](#) shows that  $\Lambda_0^2$  is fibrant in the Joyal–Kan model structure. To see that the inclusion  $\Lambda_0^2 \hookrightarrow \Delta^2$  is not a trivial Joyal–Kan cofibration, note that the lifting problem

$$\begin{array}{ccc} \Lambda_0^2 & \xlongequal{\quad} & \Lambda_0^2 \\ \downarrow & \nearrow \text{dotted} & \downarrow f \\ \Delta^2 & \xrightarrow{f} & P \end{array}$$

does not admit a solution because the inclusion of simplicial sets  $\Lambda_0^2 \hookrightarrow \Delta^2$  does not admit a retraction.

To prove the claim for  $n \geq 3$ , one can easily construct a 1-category  $C_{0,f}^n$  along with a natural inclusion  $\phi_f: \Lambda_0^n \hookrightarrow C_{0,f}^n$  that does not extend to  $\Delta^n$  as follows: adjoin a new morphism  $a: 1 \rightarrow n$  to  $\Delta^n$  so that  $a$  and the unique morphism  $1 \rightarrow n$  are equalized by the unique morphism  $0 \rightarrow 1$ , then formally adjoin inverses to all morphisms  $i \rightarrow j$  such that  $f(i) = f(j)$ . The inclusion  $\phi_f: \Lambda_0^n \hookrightarrow C_{0,f}^n$  is not the standard one, but one with the property that the edge  $\Delta^{\{1,n\}}$  is sent to the morphism  $a$ . Thus  $\phi_f$  does not extend to  $\Delta^n$ . The morphism  $f|_{\Lambda_0^n}$  extends to a stratification  $\bar{f}: C_{0,f}^n \rightarrow P$  that makes  $C_{0,f}^n$  a fibrant object in the Joyal–Kan model structure, and the inclusion  $\Lambda_0^n \hookrightarrow \Delta^n$  is not a trivial Joyal–Kan cofibration in  $\mathbf{sSet}_P$  since the lifting problem

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{\phi_f} & C_{0,f}^n \\ \downarrow & \nearrow \text{dotted} & \downarrow \bar{f} \\ \Delta^n & \xrightarrow{f} & P \end{array}$$

does not admit a solution. □

**2.3.4 Notation.** Let  $P$  be a poset. Write  $J_P \subset \text{Mor}(\mathbf{sSet}_P)$  for the set of all horn inclusions  $i: \Lambda_k^n \hookrightarrow \Delta^n$  over  $P$  that are Joyal–Kan equivalences.

We can use the set  $J_P$  to identify fibrations between fibrant objects of the Joyal–Kan model structure on  $\mathbf{sSet}_P$ . First we record a convenient fact.

**2.3.5 Lemma.** *Let  $f: X \rightarrow Y$  be a conservative functor between quasicategories. The following are equivalent:*

- (2.3.5.1) *For every equivalence  $e: y \simeq y'$  in  $Y$  and object  $\tilde{y} \in X$  such that  $f(\tilde{y}) = y$ , there exists an equivalence  $\tilde{e}: \tilde{y} \simeq \tilde{y}'$  in  $X$  such that  $f(\tilde{e}) = e$ .*

(2.3.5.2) For every equivalence  $e: y \simeq y'$  in  $Y$  and object  $\tilde{y} \in X$  such that  $f(\tilde{y}) = y$ , there exists a morphism  $\tilde{e}: \tilde{y} \rightarrow \tilde{y}'$  in  $X$  such that  $f(\tilde{e}) = e$ .

**2.3.6 Proposition.** Let  $P$  be a poset and  $f: X \rightarrow Y$  a morphism in  $s\mathbf{Set}_{/P}$  between fibrant objects in the Joyal–Kan model structure. Then the following are equivalent:

- (2.3.6.1) The morphism  $f$  is a Joyal–Kan fibration.
- (2.3.6.2) The morphism  $f$  is a Joyal fibration, equivalently, an isofibration.
- (2.3.6.3) The morphism  $f$  satisfies the right lifting property with respect to  $J_P$ .
- (2.3.6.4) The morphism  $f$  is an inner fibration and the restriction of  $f$  to each stratum is a Kan fibration.
- (2.3.6.5) The morphism  $f$  is an inner fibration and satisfies the right lifting property with respect to  $E_P$ .

*Proof.* The equivalence of (2.3.6.1) and (2.3.6.2) is immediate from the fact that the Joyal–Kan model structure is a left Bousfield localization of the Joyal model structure.

Now we show that (2.3.6.2) implies (2.3.6.3). Assume that  $f$  is an isofibration. Since  $f$  is an isofibration,  $f$  is an inner fibration, hence lifts against inner horns in  $J_P$ . Now consider the lifting problem

$$(2.3.7) \quad \begin{array}{ccc} \Delta^{\{i\}} & \xrightarrow{h} & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^1 & \xrightarrow{h'} & Y \end{array}$$

where the inclusion  $\Delta^{\{i\}} \hookrightarrow \Delta^1$  is in  $J_P$ . Since  $Y$  is fibrant in the Joyal–Kan model structure, the edge  $h'(\Delta^1)$  is an equivalence in  $Y$ . Lemma 2.3.5 (and its dual) now shows that the lifting problem (2.3.7) admits a solution. Finally, if  $n \geq 2$  and  $k = 0$  or  $k = n$ , then given a lifting problem

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{h} & X \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \xrightarrow{h'} & Y \end{array}$$

where the horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$  is in  $J_P$ , since  $X$  and  $Y$  are fibrant in the Joyal–Kan model structure:

- If  $k = 0$ , then  $h(\Delta^{\{0,1\}})$  and  $h'(\Delta^{\{0,1\}})$  are equivalences.
- If  $k = n$ , then  $h(\Delta^{\{n-1,n\}})$  and  $h'(\Delta^{\{n-1,n\}})$  are equivalences.

In either case, the desired lift exists because  $f$  is an inner fibration and the outer horn is “special” [20, Theorem 2.2; 29, p. 236].

The fact that (2.3.6.3) implies (2.3.6.4) is obvious from the identification of  $J_P$  (Proposition 2.3.1).

The fact that (2.3.6.4) implies (2.3.6.5) is obvious from the definition of  $E_p$  and the fact that the restriction of  $f$  to each stratum is a Kan fibration.

Now we show that (2.3.6.5) implies (2.3.6.2). Assume that  $f$  is an inner fibration and satisfies the right lifting property with respect to  $E_p$ . Since  $f$  is conservative (Corollary 2.2.6) and the equivalences in  $Y$  lie in individual strata, Lemma 2.3.5 combined with the fact that  $f$  satisfies the right lifting property with respect to  $E_p$  show that  $f$  is an isofibration.  $\square$

**2.3.8 Corollary.** *Let  $P$  be a poset and  $X$  an object of  $s\mathbf{Set}_{/P}$ . Then  $X$  is fibrant in the Joyal–Kan model structure if and only if the stratification  $X \rightarrow P$  satisfies the right lifting property with respect to  $J_P$ .*

## 2.4 Simpliciality of the Joyal–Kan model structure

Unlike the Kan model structure on  $s\mathbf{Set}$ , the Joyal model structure is *not* simplicial. As a result, it does not follow formally from the definition that the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  is simplicial. In this subsection we recall three criteria that guarantee that a model structure is simplicial, and verify the first two of them. We verify the third in §2.5.

**2.4.1.** By appealing to [HTT, Proposition A.3.1.7], we can prove that the Joyal–Kan model structure is simplicial by proving the following three claims:

(2.4.1.1) Given a monomorphism of simplicial sets  $j: A \rightarrowtail B$  and a Joyal–Kan cofibration  $i: X \rightarrowtail Y$  in  $s\mathbf{Set}_{/P}$ , the pushout-tensor

$$i \hat{\rtimes} j: (X \rtimes B) \sqcup^{X \rtimes A} (Y \rtimes A) \rightarrow Y \rtimes B$$

is a Joyal–Kan cofibration.

(2.4.1.2) For every  $n \geq 0$  and every object  $X \in s\mathbf{Set}_{/P}$ , the natural map

$$X \rtimes \Delta^n \rightarrow X \rtimes \Delta^0 \cong X$$

is a Joyal–Kan equivalence.

(2.4.1.3) The collection of weak equivalences in the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  is stable under filtered colimits.

Note that (2.4.1.1) follows from Remark 2.1.3 and the fact that cofibrations in the Joyal–Kan model structure are monomorphisms of simplicial sets (Proposition 2.1.4).

We first concern ourselves with (2.4.1.2). Since the natural map

$$X \rtimes \Delta^n \rightarrow X \rtimes \Delta^0 \cong X$$

admits a section  $X \cong X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^n$ , it suffices to show that this section is a trivial Joyal–Kan cofibration. In fact, we prove a more precise claim.

**2.4.2 Notation.** Let  $P$  be a poset.

- Write  $IH_P \subset J_P$  for the *inner* horn inclusions in  $J_P$ .
- Write  $LH_P \subset J_P$  for those horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  in  $J_P$  where  $n \geq 1$  and  $0 \leq k < n$ , i.e., the *left* horn inclusions in  $J_P$ .

Note that [Proposition 2.3.1](#) gives complete characterizations of  $IH_P$  and  $LH_P$ .

**2.4.3 Proposition.** *Let  $P$  be a poset and  $n \geq 0$  an integer. For any object  $X \in s\mathbf{Set}_P$ , the inclusion*

$$X \cong X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^n$$

*is in the weakly saturated class generated by  $LH_P$ , in particular, a trivial Joyal–Kan cofibration in  $s\mathbf{Set}_P$ .*

The next proposition (and its proof) is a stratified variant of [HTT, Propositions 2.1.2.6 & 3.1.1.5] which we use to prove [Proposition 2.4.3](#).

**2.4.4 Proposition.** *Let  $P$  be a poset. Consider the following classes of morphisms in  $s\mathbf{Set}_P$ :*

(2.4.4.1) *All inclusions*

$$(\partial\Delta^m \rtimes \Delta^1) \sqcup^{\partial\Delta^m \rtimes \Delta^{\{0\}}} (\Delta^m \rtimes \Delta^{\{0\}}) \hookrightarrow \Delta^m \rtimes \Delta^1,$$

*where  $m \geq 0$  and  $\Delta^m \in s\mathbf{Set}_P$  is any  $m$ -simplex over  $P$ .*

(2.4.4.2) *All inclusions*

$$(A \rtimes \Delta^1) \sqcup^{A \rtimes \Delta^{\{0\}}} (B \rtimes \Delta^{\{0\}}) \hookrightarrow B \rtimes \Delta^1,$$

*where  $A \hookrightarrow B$  is any monomorphism in  $s\mathbf{Set}_P$ .*

*The classes (2.4.4.1) and (2.4.4.2) generate the same weakly saturated class of morphisms in  $s\mathbf{Set}_P$ . Moreover, this weakly saturated class of morphisms generated by (2.4.4.1) or (2.4.4.2) is contained in the weakly saturated class of morphisms generated by  $LH_P$ .*

*Proof.* Since the inclusions  $\partial\Delta^m \hookrightarrow \Delta^m$  in  $s\mathbf{Set}_P$  generate the monomorphisms in  $s\mathbf{Set}_P$ , to see that each of the morphisms specified in (2.4.4.2) is contained in the weakly saturated class generated by (2.4.4.1), it suffices to work simplex-by-simplex with the inclusion  $A \hookrightarrow B$ . The converse is obvious since the class specified by (2.4.4.1) is contained in the class specified by (2.4.4.2).

To complete the proof, we show that for each  $P$ -stratified  $m$ -simplex  $\Delta^m \in s\mathbf{Set}_P$ , the inclusion

$$(2.4.5) \quad (\partial\Delta^m \rtimes \Delta^1) \sqcup^{\partial\Delta^m \rtimes \Delta^{\{0\}}} (\Delta^m \rtimes \Delta^{\{0\}}) \hookrightarrow \Delta^m \rtimes \Delta^1$$

belongs to the weakly saturated class generated by  $LH_P$ . The proof of this is *verbatim* the same as the proof of [HTT, Proposition 2.1.2.6], which writes the inclusion (2.4.5) as a composite of pushouts of horn inclusions, all of which are in  $LH_P$ .  $\square$

**2.4.6 Corollary.** *Let  $P$  be a poset. For any  $P$ -stratified simplicial set  $X \in s\mathbf{Set}_P$ , the inclusion  $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^1$  is in the weakly saturated class generated by  $LH_P$ .*

*Proof.* In (2.4.4.2) set  $A = \emptyset$  and  $B = X$ . □

**2.4.7 Notation.** Let  $n \geq 0$  be an integer. Write  $\text{Spn}^n \subset \Delta^n$  for the *spine* of  $\Delta^n$ , defined by

$$\text{Spn}^n := \Delta^{\{0,1\}} \cup^{\Delta^{\{1\}}} \dots \cup^{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}.$$

Now we use [Corollary 2.4.6](#) and the fact that the spine inclusion  $\text{Spn}^n \hookrightarrow \Delta^n$  is inner anodyne to address [Proposition 2.4.3](#).

**2.4.8 Lemma.** *Let  $P$  be a poset and  $n \geq 0$  an integer. For any  $P$ -stratified simplicial set  $X \in \mathbf{sSet}_{/P}$ , the inclusion  $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \text{Spn}^n$  is in the weakly saturated class generated by  $LH_P$ .*

*Proof.* Noting that  $\text{Spn}^1 = \Delta^1$ , factor the inclusion  $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \text{Spn}^n$  as a composite

$$X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^1 \hookrightarrow X \rtimes \text{Spn}^2 \hookrightarrow \dots \hookrightarrow X \rtimes \text{Spn}^n.$$

The inclusion  $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^1$  is in the weakly saturated class generated by  $LH_P$  ([Corollary 2.4.6](#)), so it suffices to show that for  $1 \leq k \leq n-1$ , the inclusion  $X \rtimes \text{Spn}^k \hookrightarrow X \rtimes \text{Spn}^{k+1}$  is in the weakly saturated class generated by  $LH_P$ . To see this, note that the inclusion  $X \rtimes \text{Spn}^k \hookrightarrow X \rtimes \text{Spn}^{k+1}$  is given by the pushout

$$\begin{array}{ccc} X \rtimes \Delta^{\{k\}} & \hookrightarrow & X \rtimes \Delta^{\{k,k+1\}} \\ \downarrow & & \downarrow \\ X \rtimes \text{Spn}^k & \hookrightarrow & X \rtimes \text{Spn}^{k+1} \end{array},$$

and by [Corollary 2.4.6](#) the inclusion  $X \rtimes \Delta^{\{k\}} \hookrightarrow X \rtimes \Delta^{\{k,k+1\}}$  is in the weakly saturated class generated by  $LH_P$ . □

*Proof of Proposition 2.4.3.* The inclusion  $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \Delta^n$  factors as a composite

$$X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \text{Spn}^n \hookrightarrow X \rtimes \Delta^n.$$

To conclude, first note that by [Lemma 2.4.8](#) the inclusion  $X \rtimes \Delta^{\{0\}} \hookrightarrow X \rtimes \text{Spn}^n$  is in the weakly saturated class generated by  $LH_P$ . Second, since the inclusion  $\text{Spn}^n \hookrightarrow \Delta^n$  is inner anodyne, the inclusion

$$X \rtimes \text{Spn}^n = X \rtimes \text{Spn}^n \hookrightarrow X \rtimes \Delta^n = X \times \Delta^n$$

is inner anodyne [[HTT](#), Corollary 2.3.2.4], hence in the weakly saturated class generated by  $LH_P$ . □

## 2.5 Stability of weak equivalences under filtered colimits

In this subsection we explain how to fibrantly replace simplicial sets over  $P$  whose strata are Kan complexes without changing their strata, and use this to deduce that Joyal–Kan equivalences between such objects are Joyal equivalences (Proposition 2.5.4). We leverage this to show that Joyal–Kan equivalences are stable under filtered colimits (Proposition 2.5.9), verifying the last criterion to show that the Joyal–Kan model structure is simplicial (Theorem 2.5.10). We deduce that the Joyal–Kan model structure presents the  $\infty$ -category  $\mathbf{Str}_P$  (Corollary 2.5.11).

**2.5.1 Notation.** Let  $P$  be a poset. Write  $IH_P^{nv} \subset IH_P$  for those inner horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  over  $P$  that are *not vertical* in the sense that the stratification  $\Delta^n \rightarrow P$  is not a constant map.

**2.5.2 Lemma.** Let  $f : X \rightarrow Y$  be a morphism in  $s\mathbf{Set}_{/P}$ . Then there exists a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & \downarrow j \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}, \end{array}$$

in  $s\mathbf{Set}_{/P}$  where:

- (2.5.2.1) The morphisms  $i$  and  $j$  are  $IH_P^{nv}$ -cell maps.
- (2.5.2.2) The morphism  $\tilde{f}$  satisfies the right lifting property with respect to  $IH_P^{nv}$ .
- (2.5.2.3) The morphisms  $i$  and  $j$  restrict to isomorphism on strata, i.e., for all  $p \in P$  the morphisms  $i$  and  $j$  restrict to isomorphisms of simplicial sets  $i : X_p \xrightarrow{\sim} \tilde{X}_p$  and  $j : Y_p \xrightarrow{\sim} \tilde{Y}_p$ .
- (2.5.2.4) If, in addition, all of the strata of  $X$  and  $Y$  are quasicategories, then  $\tilde{X}$  and  $\tilde{Y}$  can be chosen to be quasicategories.

In particular, if all of the strata of  $X$  and  $Y$  are Kan complexes, then  $\tilde{X}$  and  $\tilde{Y}$  can be chosen to be fibrant in the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$ .

*Proof.* Since the morphisms in  $IH_P^{nv}$  all have small domains, we can apply the small object argument to construct a square

$$(2.5.3) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & \downarrow j \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}, \end{array}$$

where  $i$  and  $j$  are  $IH_P^{nv}$ -cell maps and  $\tilde{f}$  has the right lifting property with respect to  $IH_P^{nv}$ , which proves (2.5.2.1) and (2.5.2.2).

To prove (2.5.2.3) we examine the constructions of  $\tilde{X}$  and  $\tilde{Y}$  via the small object argument. Both morphisms  $i$  and  $j$  are obtained by a transfinite composite of pushouts of inner horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  in  $IH_P^{nv}$ . Hence to prove (2.5.2.3) it suffices to show that given an object  $A \in \mathbf{sSet}/_P$  and a morphism  $f: \Lambda_k^n \rightarrow A$ , where  $\Lambda_k^n \in \mathbf{sSet}/_P$  is the domain of a morphism  $g: \Lambda_k^n \hookrightarrow \Delta^n$  in  $IH_P^{nv}$ , the morphism  $\bar{g}$  in the pushout square

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{f} & A \\ g \downarrow & & \downarrow \bar{g} \\ \Delta^n & \xrightarrow{\bar{f}} & A' \end{array}$$

induces an isomorphism (of simplicial sets) on strata. To see this, let  $\sigma: \Delta^n \rightarrow P$  denote the stratification of the target of  $g$ . Since  $g \in IH_P^{nv}$ , the stratification  $\sigma$  is not a constant map. We claim that for all  $p \in P$  and  $m \geq 0$ , the  $m$ -simplices of  $A_p \subset A'_p$  and  $A'_p$  coincide. If  $p \notin \sigma(\Delta^n)$  or  $m < n - 1$ , this is obvious. Let us consider the remaining cases.

- If  $m = n - 1$ , then note that the only additional  $(n - 1)$ -simplex adjoined to  $A$  in the pushout defining  $A'$  is the image of the face  $\Delta^{\{0, \dots, k-1, k+1, \dots, n\}} \subset \Delta^n$ . Since the horn  $\Lambda_k^n \subset \Delta^n$  is an inner horn, both vertices  $\Delta^{\{0\}}$  and  $\Delta^{\{n\}}$  are contained in  $\Delta^{\{0, \dots, k-1, k+1, \dots, n\}}$ . Since the stratification  $\sigma: \Delta^n \rightarrow P$  is not constant, the image of  $\Delta^{\{0, \dots, k-1, k+1, \dots, n\}}$  in  $A'$  intersects more than one stratum. Hence for each  $p \in P$ , the  $(n - 1)$ -simplices of the strata  $A_p$  and  $A'_p$  coincide.
- If  $m = n$ , then note that the only additional nondegenerate  $n$ -simplex adjoined to  $A$  in the pushout defining  $A'$  is the unique nondegenerate  $n$ -simplex of  $\Delta^n$ . Since the stratification  $\sigma: \Delta^n \rightarrow P$  is non-constant, the image of this top-dimensional simplex under  $\bar{f}$  intersects more than one stratum. Similarly, note that since the image of the face  $\Delta^{\{0, \dots, k-1, k+1, \dots, n\}} \subset \Delta^n$  in  $A'$  intersects more than one stratum (by the previous point), all of its degeneracies intersect more than one stratum. But the image of  $\Delta^n$  and images of the degeneracies of  $\Delta^{\{0, \dots, k-1, k+1, \dots, n\}}$  under  $\bar{f}$  are the only  $n$ -simplices adjoined to  $A$  in the pushout defining  $A'$ . Hence for each  $p \in P$ , the  $n$ -simplices of the strata  $A_p$  and  $A'_p$  coincide.
- If  $m > n$ , then the claim follows from the fact that the  $m$ -simplices of  $A_p$  and  $A'_p$  coincide for all  $m \leq n$  and the  $n$ -skeletality of  $\Delta^n$ .

Now we prove (2.5.2.4); assume that the strata of  $X$  and  $Y$  are quasicategories. To see that  $\tilde{Y}$  is a quasicategory, note that by the construction of the factorization (2.5.3) via the small object argument,  $\tilde{Y}$  is given by factoring the unique morphism  $Y \rightarrow P$  to the final object as a composite

$$Y \xrightarrow{j} \tilde{Y} \xrightarrow{h} P$$

of the  $IH_P^{nv}$ -cell map  $j$  followed by a morphism  $h$  with the right lifting property with respect to  $IH_P^{nv}$ . To show that  $\tilde{Y}$  is a quasicategory, we prove that  $h$  is an inner fibration. By the definition of  $IH_P^{nv}$ , the morphism  $h$  lifts against all inner horns  $\Lambda_k^n \hookrightarrow \Delta^n$  over  $P$  where the stratification of  $\Delta^n$  is not constant. Thus to check that  $h: \tilde{Y} \rightarrow P$  is an



inner fibration, it suffices to check that for every inner horn  $\Lambda_k^n \hookrightarrow \Delta^n$  over  $P$  where the stratification  $\sigma: \Delta^n \rightarrow P$  is constant at a vertex  $p \in P$ , every lifting problem

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & \tilde{Y} \\ \downarrow & \nearrow & \downarrow h \\ \Delta^n & \xrightarrow{\sigma} & P \end{array}$$

admits a solution. The desired lift exists by (2.5.2.3) because the stratum  $\tilde{Y}_p \cong Y_p$  is a quasicategory by assumption.

We conclude that  $\tilde{X}$  is a quasicategory by showing that the stratification  $\tilde{X} \rightarrow P$  is an inner fibration. First, note that since  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$  and  $h: \tilde{Y} \rightarrow P$  have the right lifting property with respect to  $IH_P^{nv}$ , so does the stratification  $h\tilde{f}: \tilde{X} \rightarrow P$ . Hence to show that the stratification  $\tilde{X} \rightarrow P$  is an inner fibration, it suffices to show that  $\tilde{X} \rightarrow P$  lifts against inner horns  $\Lambda_k^n \hookrightarrow \Delta^n$  over  $P$  where the stratification  $\sigma: \Delta^n \rightarrow P$  is constant. Again, the desired lift exists by (2.5.2.3) because the strata of  $\tilde{X}$  are quasicategories.  $\square$

**2.5.4 Proposition.** *Let  $P$  be a poset and  $f: X \rightarrow Y$  a morphism in  $s\mathbf{Set}_{/P}$ . If the strata of  $X$  and  $Y$  are all Kan complexes, then  $f$  is a Joyal–Kan equivalence if and only if  $f$  is a Joyal equivalence.*

*Proof.* Since the Joyal–Kan equivalences between fibrant objects of the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  are precisely the Joyal equivalences (Corollary 2.2.5), by 2-of-3 it suffices to show that there exists a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \downarrow & & \downarrow j \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \end{array}$$

in  $s\mathbf{Set}_{/P}$ , where  $\tilde{X}$  and  $\tilde{Y}$  are Joyal–Kan fibrant objects, and  $i: X \simeq \tilde{X}$  and  $j: Y \simeq \tilde{Y}$  are Joyal equivalences. This follows from Lemma 2.5.2 since  $IH_P^{nv}$ -cell maps are, in particular, Joyal equivalences.  $\square$

**2.5.5.** Since Joyal equivalences are stable under filtered colimits [HTT, Theorem 2.2.5.1 & p. 90], to show that Joyal–Kan equivalences are stable under filtered colimits, Proposition 2.5.4 reduces us to constructing a functor  $F: s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}_{/P}$  that lands in strata-wise Kan complexes, admits a natural weak equivalence  $\mathrm{id}_{s\mathbf{Set}_{/P}} \simeq F$ , and preserves filtered colimits. We accomplish this by applying Kan’s  $\mathrm{Ex}^\infty$  functor vertically to each stratum.

**2.5.6 Construction.** Let  $P$  be a poset. Define a functor  $\mathrm{VEx}_P^\infty: s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}_{/P}$  by the assignment

$$X \mapsto X \sqcup^{\mathrm{Obj}(P) \times_P X} \mathrm{Ex}^\infty(\mathrm{Obj}(P) \times_P X) \cong X \sqcup^{\coprod_{p \in P} X_p} \left( \coprod_{p \in P} \mathrm{Ex}^\infty(X_p) \right),$$

where the pushout is taken in  $s\mathbf{Set}_{/P}$ , and the two stratifications  $\coprod_{p \in P} \mathrm{Ex}^\infty(X_p) \rightarrow P$  and  $\coprod_{p \in P} X_p \rightarrow P$  are induced by the constant maps  $\mathrm{Ex}^\infty(X_p) \rightarrow P$  and  $X_p \rightarrow P$  at  $p \in P$ .

We claim that the natural inclusion

$$X \hookrightarrow \mathrm{VEx}_P^\infty(X)$$

is a trivial Joyal–Kan cofibration. To see this, observe that for each  $p \in P$ , the inclusion  $X_p \hookrightarrow \mathrm{Ex}^\infty(X_p)$  is a trivial Kan cofibration, so in the weakly saturated class generated by the horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  in  $s\mathbf{Set}$ , where  $n \geq 0$  and  $0 \leq k \leq n$ . Thus, stratifying  $X_p$  and  $\mathrm{Ex}^\infty(X_p)$  via the constant maps at  $p \in P$ , by [Proposition 2.3.1](#) the inclusion  $X_p \hookrightarrow \mathrm{Ex}^\infty(X_p)$  is a trivial Joyal–Kan cofibration in  $s\mathbf{Set}_{/P}$ . To conclude, note that by definition the inclusion  $X \hookrightarrow \mathrm{VEx}_P^\infty(X)$  is a pushout of the trivial Joyal–Kan cofibration

$$\coprod_{p \in P} X_p \hookrightarrow \coprod_{p \in P} \mathrm{Ex}^\infty(X_p).$$

In particular, note that by [Proposition 2.5.4](#) a morphism  $f: X \rightarrow Y$  in  $s\mathbf{Set}_{/P}$  is a Joyal–Kan equivalence if and only if

$$\mathrm{VEx}_P^\infty(f): \mathrm{VEx}_P^\infty(X) \rightarrow \mathrm{VEx}_P^\infty(Y)$$

is a Joyal equivalence.

**2.5.7 Warning.** The functor  $\mathrm{VEx}_P^\infty$  in general does *not* preserve quasicategories over  $P$  and is not a fibrant replacement for the Joyal–Kan model structure unless  $P$  is discrete.

**2.5.8 Lemma.** *Let  $P$  be a poset. Then the functor  $\mathrm{VEx}_P^\infty: s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}_{/P}$  preserves filtered colimits.*

*Proof.* First, since filtered colimits commute with finite limits in  $s\mathbf{Set}$ , the functor

$$\mathrm{Obj}(P) \times_P -: s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}_{/P}$$

preserves filtered colimits. Second, since Kan’s  $\mathrm{Ex}^\infty$  functor preserves filtered colimits, the functor  $s\mathbf{Set}_{/P} \rightarrow s\mathbf{Set}_{/P}$  given by the assignment

$$X \mapsto \mathrm{Ex}^\infty(\mathrm{Obj}(P) \times_P X) \cong \coprod_{p \in P} \mathrm{Ex}^\infty(X_p)$$

preserves filtered colimits. The claim is now clear from the definition of  $\mathrm{VEx}_P^\infty$ .  $\square$

Combining our observation (2.5.5) with [Lemma 2.5.8](#) we deduce:

**2.5.9 Proposition.** *For any poset  $P$ , Joyal–Kan equivalences in  $s\mathbf{Set}_{/P}$  are stable under filtered colimits.*

[Propositions 2.4.3](#) and [2.5.9](#) and [Remark 2.1.3](#) verify the three conditions of (2.4.1) proving:

**2.5.10 Theorem.** *For any poset  $P$ , the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  is simplicial.*

From this we immediately deduce that the Joyal–Kan model structure presents  $\mathbf{Str}_P$ .

**2.5.11 Corollary.** *Let  $P$  be a poset. Then the underlying quasicategory of the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  is the quasicategory  $\mathbf{Str}_P$  of abstract  $P$ -stratified homotopy types.*

### 3 The Joyal–Kan model structure & stratified topological spaces

In this section we explain the interaction between  $P$ -stratified topological spaces and the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  via Lurie’s exit-path construction.

#### 3.1 Elementary results

**3.1.1 Corollary.** *Let  $P$  be a poset and  $T$  a  $P$ -stratified topological space. Then the  $P$ -stratified simplicial set  $\mathrm{Sing}_P(T)$  is a Joyal–Kan fibrant object of  $s\mathbf{Set}_{/P}$  if and only if  $\mathrm{Sing}_P(T)$  is a quasicategory.*

*Proof.* Combine (1.2.7) and Proposition 2.2.4. □

**3.1.2.** If  $f : T \rightarrow U$  is a morphism in  $\mathbf{Top}_{/P}$  and both  $\mathrm{Sing}_P(T)$  and  $\mathrm{Sing}_P(U)$  are quasicategories, then  $f$  is a weak equivalence in the Douteau–Henriques model structure on  $\mathbf{Top}_{/P}$  if and only if the morphism  $\mathrm{Sing}_P(f)$  is a weak equivalence in the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  (1.3.9).

Since the exit-path simplicial set of a conically stratified topological space is a quasicategory (1.2.8), we deduce:

**3.1.3 Corollary.** *Let  $P$  be a poset. If  $T \in \mathbf{Top}_{/P}$  is conically stratified, then the simplicial set  $\mathrm{Sing}_P(T)$  is a fibrant object in the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$ .*

Not all stratified topological spaces — even those arising as geometric realizations of quasicategories — are conically stratified.

**3.1.4 Example.** Stratify the quasicategory  $\Lambda_0^2$  over  $[1]$  via the map sending 0 and 1 to 0 and 2 to 1. The  $[1]$ -stratified topological space  $|\Lambda_0^2|_{[1]}$  is not conically stratified. Moreover, the  $[1]$ -stratified simplicial set  $\mathrm{Sing}^{[1]}|\Lambda_0^2|_{[1]}$  is not a quasicategory.

**3.1.5 Warning.** Example 3.1.4 shows that unlike the Kan model structure (i.e., the Joyal–Kan model structure where  $P = *$ ), if  $P$  is a non-discrete poset, the functor  $\mathrm{Sing}_P|-|_P$  is not a fibrant replacement for the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$ .

Since the Joyal–Kan model structure on  $s\mathbf{Set}_{/P}$  presents the  $\infty$ -category  $\mathbf{Str}_P$  (Corollary 2.5.11), we deduce the following variant of a result of Miller [25, Theorem 6.10; 26, Theorem 6.3].

**3.1.6 Corollary.** *Let  $P$  be a poset and let  $f : T \rightarrow U$  be a morphism in  $\mathbf{Top}_{/P}$ . If  $\mathrm{Sing}_P(T)$  and  $\mathrm{Sing}_P(U)$  are quasicategories, then the morphism  $\mathrm{Sing}_P(f)$  is an equivalence in when regarded as a morphism in  $\mathbf{Str}_P$  if and only if the following conditions are satisfied:*

(3.1.6.1) *For each  $p \in P$ , the induced map on strata  $T_p \rightarrow U_p$  is a weak homotopy equivalence of topological spaces.*

(3.1.6.2) *For all  $p, q \in P$  with  $p < q$ , the induced map on topological links*

$$\mathrm{Map}_{\mathbf{Top}_{/P}}(|\{p < q\}|_P, T) \rightarrow \mathrm{Map}_{\mathbf{Top}_{/P}}(|\{p < q\}|_P, U)$$

*is a weak homotopy equivalence of topological spaces.*

*Proof.* Combine (1.2.7), (1.1.9), and Remark 1.2.9. □

### 3.2 Relation to other homotopy theories of stratified spaces

Now we use the Joyal–Kan model structure to compare our homotopy theory  $\mathbf{Str}_P$  of  $P$ -stratified spaces to other existing homotopy theories. The takeaway is that our homotopy theory subsumes all others. Throughout this subsection  $P$  denotes a poset.

**3.2.1 Comparison** (conically smooth stratified spaces). In work with Tanaka [2, §3], Ayala and Francis introduced *conically smooth stratified spaces* which they further studied in work with Rozenblyum [1]. Write  $\mathbf{Con}_P$  for their category of conically smooth stratified spaces and conically smooth maps, stratified by a fixed poset  $P$ . Note that conically smooth stratified spaces are, in particular, conically stratified. The Ayala–Francis–Rozenblyum  $\infty$ -category of  $P$ -stratified spaces is the  $\infty$ -category obtained from  $\mathbf{Con}_P$  by inverting the class  $H$  of stratified homotopy equivalences [1, Theorem 2.4.5]. The functor  $\mathrm{Sing}_P : \mathbf{Con}_P \rightarrow s\mathbf{Set}_{/P}$  sends the class  $H$  to Joyal–Kan equivalences, hence induces a functor of  $\infty$ -categories  $\mathrm{Sing}_P : \mathbf{Con}_P[H^{-1}] \rightarrow \mathbf{Str}_P$ . As a result of [1, Lemma 3.3.9 & Theorem 4.2.8] the functor  $\mathrm{Sing}_P : \mathbf{Con}_P[H^{-1}] \rightarrow \mathbf{Str}_P$  is fully faithful.

**3.2.2 Comparison** (stratified spaces with exit path  $\infty$ -categories). Consider the full subcategory  $\mathbf{Top}_{/P}^{\mathrm{ex}} \subset \mathbf{Top}_{/P}$  of those  $P$ -stratified topological spaces  $T$  for which the simplicial set  $\mathrm{Sing}_P(T)$  is a quasicategory. Note that in particular,  $\mathbf{Top}_{/P}^{\mathrm{ex}}$  contains all conically  $P$ -stratified topological spaces (1.2.8).

A morphism  $f$  in  $\mathbf{Top}_{/P}^{\mathrm{ex}}$  is a Douteau–Henriques weak equivalence if and only if  $\mathrm{Sing}_P(f)$  is a Joyal–Kan equivalence (3.1.2). Moreover, for any  $T \in \mathbf{Top}_{/P}^{\mathrm{ex}}$ , the simplicial presheaf  $D_P(T) \cong N_P \mathrm{Sing}_P(T)$  already satisfies the Segal condition for décollages. Thus if we let  $W \subset \mathrm{Mor}(\mathbf{Top}_{/P}^{\mathrm{ex}})$  denote the class of morphisms that are sent to equivalences in the Joyal–Kan model structure under  $\mathrm{Sing}_P$ , then Theorem 1.3.10 implies that the induced functor of  $\infty$ -categories  $\mathrm{Sing}_P : \mathbf{Top}_{/P}^{\mathrm{ex}}[W^{-1}] \rightarrow \mathbf{Str}_P$  is fully faithful.

Now let us show that  $\mathrm{Sing}_P$  is essentially surjective. An object  $T$  of  $\mathbf{Top}_{/P}$  is fibrant in left model category given by the left Bousfield localization of the Douteau–Henriques model structure on  $\mathbf{Top}_{/P}$  that presents  $\mathbf{D}\acute{e}c_P$  if and only if  $D_P(T) \in \mathrm{Fun}(\mathrm{sd}(P)^{\mathrm{op}}, s\mathbf{Set})$  is projective fibrant and satisfies the Segal condition for décollages. Since the simplicial presheaf  $D_P(T) \cong N_P \mathrm{Sing}_P(T)$  is always projective fibrant, the requirement that  $D_P(T)$  satisfy the Segal condition says exactly that  $\mathrm{Sing}_P(T)$  is a quasicategory. We conclude that  $\mathrm{Sing}_P : \mathbf{Top}_{/P}^{\mathrm{ex}}[W^{-1}] \hookrightarrow \mathbf{Str}_P$  is essentially surjective (1.3.11), hence an equivalence of  $\infty$ -categories.

From Comparison 3.2.1 we deduce that we have a commutative triangle of fully faithful functors of  $\infty$ -categories

$$\begin{array}{ccc} \mathbf{Con}_P[H^{-1}] & \xrightarrow{\mathrm{Sing}_P} & \mathbf{Str}_P \\ \downarrow & \nearrow \sim & \\ \mathbf{Top}_{/P}^{\mathrm{ex}}[W^{-1}] & \xrightarrow{\mathrm{Sing}_P} & \end{array},$$

where the vertical functor is induced by the functor  $\mathbf{Con}_P \rightarrow \mathbf{Top}_{/P}^{\mathrm{ex}}$  forgetting conically smooth structures. In particular, the theory of conically stratified spaces with equivalences on exit-path  $\infty$ -categories inverted subsumes the Ayala–Francis–Rozenblyum

theory of stratified spaces. Importantly, the former theory contains all topologically stratified spaces and Whitney stratified spaces. On the other hand, it is not known if all Whitney stratified spaces can be given conically smooth structures [1, Conjecture 0.0.7].

In his thesis, Nand-Lal proved that the Joyal–Kan model structure right-transfers along the functor  $\text{Sing}_P$  to define  $s\mathbf{Set}^{\text{Joy}}$ -enriched model structure on the full subcategory of  $\mathbf{Top}_P^{\text{ex}}$  [27, Theorem 8.2.3.2].<sup>4</sup> It is natural to try to extend Nand-Lal’s model structure to all of  $\mathbf{Top}_P$  and show that the resulting adjunction is an equivalence of homotopy theories with  $\mathbf{Str}_P$ . Unfortunately, since Joyal equivalences between simplicial sets that are not quasicategories are incredibly inexplicit, doing so is exceedingly difficult. The many difficulties in trying to extend this model structure are surveyed in [27, §8.4].

## A The model structure on $\mathbf{Top}_P$ , d’après Douteau

The purpose of this appendix is to present a proof of the main result of Chapter 7 of Douteau’s thesis [8], which realizes pioneering ideas of Henriques [18]. Namely, we prove that the projective model structure on  $\text{Fun}(\text{sd}(P)^{\text{op}}, s\mathbf{Set})$  right-transfers to  $\mathbf{Top}_P$  along the adjunction of [Notation 1.3.1](#), and that the resulting Quillen adjunction is a Quillen equivalence. We follow Douteau’s general narrative, though our proofs tend to be rather different (in particular, we do not make use of Douteau’s ‘filtered homotopy groups’). We also refine a number of points; namely we refine the Quillen equivalence to a simplicial Quillen equivalence.

The proof is rather involved, so we now provide an outline of the argument. First, there is an intermediary category that will be relevant to the story: the adjunction that we are interested in factors

$$\text{Fun}(\text{sd}(P)^{\text{op}}, s\mathbf{Set}) \rightleftarrows \mathbf{Top}_{/|P|} \rightleftarrows \mathbf{Top}_P$$

through that category of topological spaces over the geometric realization of  $P$  (see [Notation A.2.1](#)). A very general technique supplied by the Transfer Theorem of Hess–Kędziorek–Riehl–Shiely [13, Corollary 2.7; 19, Corollary 3.3.4] will enable us to right-transfer the projective model structure to *both*  $\mathbf{Top}_{/|P|}$  and  $\mathbf{Top}_P$  ([Corollary A.2.9](#)). The fact that the functor  $\text{Sing}_P |P| \rightarrow P$  is a equivalence of  $\infty$ -categories easily implies that the right-hand adjunction is a Quillen equivalence ([Corollary A.3.3](#)). Thus to prove that the long adjunction is a Quillen equivalence, it suffices to prove that the left-hand adjunction is a Quillen equivalence. This is a much easier problem because of the extreme rigidity of the spaces of sections  $|\Sigma| \rightarrow T$  over  $|P|$  ([Proposition A.4.7](#)).

### A.1 Conventions on right-transferred model structures

We begin by setting our conventions for right-transferred model structures, and record a convenient reformulation of the Hess–Kędziorek–Riehl–Shiely Transfer Theorem ([Lemma A.1.4](#)).

---

<sup>4</sup>Nand-Lal uses a different convenient category of topological spaces, but of course his proofs work equally well for numerically generated spaces.

**A.1.1 Definition.** Let  $\mathcal{M}$  be a model category,  $\mathcal{N}$  a category with all limits and colimits, and  $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$  an adjunction. We say that a morphism  $f$  in  $\mathcal{N}$  is:

(A.1.1.1) A *G-fibration* if  $G(f)$  is a fibration in  $\mathcal{M}$ .

(A.1.1.2) A *G-weak equivalence* if  $G(f)$  is a weak equivalence in  $\mathcal{M}$ .

(A.1.1.3) A *G-cofibration* if  $f$  satisfies the left lifting property with respect to all morphisms that are both a *G-fibration* and a *G-weak equivalence*.

We say that the model structure on  $\mathcal{M}$  *right-transfers* to  $\mathcal{N}$  if the *G-fibrations*, *G-cofibrations*, and *G-weak equivalences* define a model structure on  $\mathcal{N}$ .

The following observations are immediate from the definitions. We will use all of them throughout the course of our proof.

**A.1.2.** Let  $\mathcal{M}$  be a model category,  $\mathcal{N}$  a category with all limits and colimits, and

$$(A.1.3) \quad F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

an adjunction. Assume that the model structure on  $\mathcal{M}$  right-transfers to  $\mathcal{N}$ . Then:

(A.1.2.1) The adjunction (A.1.3) is a Quillen adjunction.

(A.1.2.2) If for every cofibrant object  $X \in \mathcal{M}$ , the unit  $X \rightarrow GF(X)$  is a weak equivalence in  $\mathcal{M}$ , then the Quillen adjunction (A.1.3) is a Quillen equivalence.

(A.1.2.3) Given an adjunction  $\mathcal{N} \rightleftarrows \mathcal{N}'$  where  $\mathcal{N}'$  has all limits and colimits, the model structure on  $\mathcal{N}$  right-transfers to  $\mathcal{N}'$  if and only if the model structure on  $\mathcal{M}$  right-transfers to  $\mathcal{N}'$ . If these model structures right-transfer to  $\mathcal{N}'$ , then they coincide.

(A.1.2.4) Let  $\mathcal{S}$  be a monoidal model category. Assume that  $\mathcal{M}$  is a  $\mathcal{S}$ -enriched model category,  $\mathcal{N}$  admits the structure of an  $\mathcal{S}$ -enriched category making the adjunction (A.1.3) an  $\mathcal{S}$ -enriched adjunction,  $\mathcal{N}$  is tensored and cotensored over  $\mathcal{S}$ , and the right adjoint  $G$  sends the cotensoring of  $\mathcal{N}$  to the cotensoring of  $\mathcal{M}$ . Then the right-transferred model structure and the  $\mathcal{S}$ -enrichment on  $\mathcal{N}$  are compatible and make  $\mathcal{N}$  into an  $\mathcal{S}$ -enriched model category.

(A.1.2.5) If the model structure on  $\mathcal{M}$  is cofibrantly generated and  $F$  preserves small objects (e.g., if  $G$  preserves  $\kappa$ -filtered colimits for some regular cardinal  $\kappa$ ), then the right-transferred model structure on  $\mathcal{N}$  is cofibrantly generated.

The following is a convenient reformulation of the Hess–Kędziorek–Riehl–Shipley Transfer Theorem:

**A.1.4 Lemma.** *Let  $\mathcal{M}$  be an accessible model category<sup>5</sup>,  $\mathcal{N}$  a presentable category, and  $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$  an adjunction. Then the right-transferred model structure on  $\mathcal{N}$  exists if and only if the following condition is satisfied: for every morphism  $f$  of  $\mathcal{N}$ , there exists a factorization  $f = qi$ , where  $i$  is a *G-weak equivalence* and  $q$  is a *G-fibration*.*

<sup>5</sup>See [19, §3.1] for the basics on accessible model structures. For the purposes of our work it suffices to know that combinatorial model structures are accessible [19, Corollary 3.1.7].

*Proof.* By the Hess–Kędziorek–Riehl–Shipley Transfer Theorem [13, Corollary 2.7; 19, Corollary 3.3.4], it suffices to show that if a morphism  $f : X \rightarrow Y$  in  $\mathbf{N}$  has the left lifting property with respect to every  $G$ -fibration, then  $f$  is a  $G$ -weak equivalence. Choose a factorization of  $f$  as

$$f : X \xrightarrow{i} X' \xrightarrow{q} Y,$$

where  $i$  is a  $G$ -weak equivalence and  $q$  is a  $G$ -fibration. In the square

$$(A.1.5) \quad \begin{array}{ccc} X & \xrightarrow{i} & X' \\ f \downarrow & \nearrow \text{dotted} & \downarrow q \\ Y & \xlongequal{\quad} & Y \end{array},$$

a dotted lift exists because  $q$  is a  $G$ -fibration and  $f$  satisfies the left lifting property with respect to  $G$ -fibrations. Since  $i$  and  $\text{id}_Y$  are  $G$ -weak equivalences, by the 2-of-6 property all of the morphisms in (A.1.5) are  $G$ -weak equivalences.  $\square$

## A.2 Transferring the projective model structure on $\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})$

The goal of this subsection is to prove that the projective model structure on the functor category  $\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})$  right-transfers to the categories  $\mathbf{Top}_{/|P|}$  and  $\mathbf{Top}_{/P}$ . The bulk of the work is in showing that the hypotheses of Lemma A.1.4 are satisfied. We begin by fixing some notation and explaining the relationship between the different categories appearing in the proof.

**A.2.1 Notation.** Let  $P$  be a poset, write  $\pi_P : |P| \rightarrow P$  for the natural stratification (Recollection 1.2.6), and  $\hat{\pi}_P : P \rightarrow \text{Sing } |P|$  for its adjoint morphism. Then we have a chain of adjunctions

(A.2.2)

$$\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet}) \xrightleftharpoons[N_P]{L_P} \mathbf{sSet}_{/P} \xrightleftharpoons[\hat{\pi}_P^*]{\hat{\pi}_{P,!}} \mathbf{sSet}_{/\text{Sing } |P|} \xrightleftharpoons[\text{Sing}]{\quad} \mathbf{Top}_{/|P|} \xrightleftharpoons[\pi_P^*]{\pi_{P,!}} \mathbf{Top}_{/P}$$

where:

- The left adjoints  $\pi_{P,!}$  and  $\hat{\pi}_{P,!}$  denote the forgetful functors and their right adjoints  $\pi_P^*$  and  $\hat{\pi}_P^*$  are given by pullback along  $\pi_{P,!}$  and  $\hat{\pi}_{P,!}$ , respectively.
- The left adjoint  $\mathbf{sSet}_{/\text{Sing } |P|} \rightarrow \mathbf{Top}_{/|P|}$  is given by applying geometric realization and then composing with the counit  $|\text{Sing } |P|| \rightarrow |P|$ .
- The adjoint functors  $L_P$  and  $N_P$  are defined in Notation 1.3.1.

The composite left adjoint  $\mathbf{sSet}_{/P} \rightarrow \mathbf{Top}_{/|P|}$  is simply geometric realization, the composite  $\mathbf{sSet}_{/P} \rightarrow \mathbf{Top}_{/P}$  is the functor denoted by  $|-|_P$  in Recollection 1.2.6, with right adjoint  $\text{Sing}_P$ . We write

$$D_P : \mathbf{Top}_{/P} \rightarrow \text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet}) \quad \text{and} \quad D_{|P|} : \mathbf{Top}_{/|P|} \rightarrow \text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})$$

for the composite right adjoints, given by

$$T \mapsto [\Sigma \mapsto \text{Sing Map}_{/p}(|\Sigma|_p, T)] \quad \text{and} \quad T' \mapsto [\Sigma \mapsto \text{Sing Map}_{/|p|}(|\Sigma|, T')],$$

respectively (see [Remark 1.2.9](#)).

**A.2.3.** Note that all of the adjunctions appearing in [\(A.2.2\)](#) are  $s\mathbf{Set}$ -enriched, where all overcategories and functor categories have the enrichments induced by the usual  $s\mathbf{Set}$ -enrichments of  $s\mathbf{Set}$  and  $\mathbf{Top}$ . Moreover, all of the categories appearing in [\(A.2.2\)](#) have natural (co)tensorings over  $s\mathbf{Set}$  induced by the (co)tensorings of  $s\mathbf{Set}$  and  $\mathbf{Top}$  over  $s\mathbf{Set}$ . All of the right adjoints in [\(A.2.2\)](#) preserve the cotensorings over  $s\mathbf{Set}$ .

Now we explicitly construct the factorizations necessary to apply [Lemma A.1.4](#) using a relative version of a mapping path space construction.

**A.2.4 Construction** ([8, Lemme 7.2.3]). Let  $B$  be a topological space, and  $f : T \rightarrow U$  a morphism in  $\mathbf{Top}_{/B}$ . Write  $s_T : T \rightarrow B$  and  $s_U : U \rightarrow B$  for the structure morphisms, and  $c_T : T \rightarrow \text{Map}([0, 1], B)$  for the map defined by  $x \mapsto [t \mapsto s_T(x)]$ . Define a topological space  $M_B(f)$  as the pullback

$$\begin{array}{ccc} M_B(f) & \xrightarrow{\text{pr}_2} & \text{Map}([0, 1], U) \\ \text{pr}_1 \downarrow & \lrcorner & \downarrow (\text{ev}_0, s_{U,*}) \\ T & \xrightarrow{(f, c_T)} & U \times \text{Map}([0, 1], B). \end{array}$$

We regard  $M_B(f)$  as an object of  $\mathbf{Top}_{/B}$  via the composite  $s_T \text{pr}_1 : M_B(f) \rightarrow B$ .

We write  $q_f : M_B(f) \rightarrow U$  for the composite

$$M_B(f) \xrightarrow{\text{pr}_2} \text{Map}([0, 1], U) \xrightarrow{\text{ev}_1} U.$$

Let  $i_f : T \hookrightarrow M_B(f)$  denote the subspace inclusion induced by the commutative square

$$\begin{array}{ccc} T & \xrightarrow{c_f} & \text{Map}([0, 1], U) \\ \parallel & & \downarrow (\text{ev}_0, s_{U,*}) \\ T & \xrightarrow{(f, c_T)} & U \times \text{Map}([0, 1], B), \end{array}$$

where  $c_f$  is the map defined by  $x \mapsto [t \mapsto f(x)]$ . By construction  $f = q_f i_f$ ,  $\text{pr}_1 i_f = \text{id}_T$ , and the square

$$\begin{array}{ccc} M_B(f) & \xrightarrow{q_f} & U \\ \text{pr}_1 \downarrow & & \downarrow s_U \\ T & \xrightarrow{s_T} & B \end{array}$$

commutes.



**A.2.5.** In the setting of [Construction A.2.4](#), note that we have a deformation retraction

$$\begin{aligned} M_B(f) \rtimes [0, 1] &\rightarrow M_B(f) \\ ((x, \gamma), s) &\mapsto (x, [t \mapsto \gamma(st)]) \end{aligned}$$

from  $i_f \text{pr}_1$  to  $\text{id}_{M_B(f)}$  over  $B$ .

**A.2.6 Lemma.** Let  $P$  be a poset and let  $B \in \mathbf{Top}$  denote either  $P$  or  $|P|$ . Let  $f : T \rightarrow U$  a morphism in  $\mathbf{Top}_{/B}$ . Then  $f$  admits a factorization  $f = qi$ , where  $i$  is a  $D_B$ -equivalence and  $q$  is a  $D_B$ -fibration.

*Proof.* We use [Construction A.2.4](#) to factor  $f$  as  $f = q_f i_f$ . The existence of the deformation retraction [\(A.2.5\)](#) implies that  $i_f$  is a  $D_B$ -equivalence.

We prove that  $q_f$  is a  $D_B$ -fibration. Let  $\Sigma \subset P$  be a string and  $j : \Lambda_k^n \hookrightarrow \Delta^n$  a horn inclusion. We need to show that in the diagram

$$(A.2.7) \quad \begin{array}{ccc} |\Sigma|_B \rtimes |\Lambda_k^n| & \xrightarrow{g} & M_B(f) \\ \text{id} \rtimes |j| \downarrow & \nearrow \tilde{h} & \downarrow q_f \\ |\Sigma|_B \rtimes |\Delta^n| & \xrightarrow{h} & U, \end{array}$$

admits a dotted filler. Note that the inclusion  $|j| : |\Lambda_k^n| \hookrightarrow |\Delta^n|$  admits a retraction  $r' : |\Delta^n| \twoheadrightarrow |\Lambda_k^n|$  and a homotopy  $H' : |\Delta^n| \times [0, 1] \rightarrow |\Delta^n|$  from  $|j| r'$  to  $\text{id}_{|\Delta^n|}$  fixing  $|\Lambda_k^n|$  (see [\[11, §4.5\]](#)). Let  $d' : |\Delta^n| \rightarrow [0, 1]$  be a map such that  $(d')^{-1}(0) = |\Lambda_k^n|$ . Now define

$$r := \text{id}_{|\Sigma|_B} \rtimes r' \quad \text{and} \quad H := \text{id}_{|\Sigma|_B} \rtimes H'.$$

Write  $d$  for the composite of the projection  $\text{pr}_2 : |\Sigma|_B \rtimes |\Delta^n| \rightarrow |\Delta^n|$  with  $d'$ . Then  $r$  is a retraction of  $\text{id}_{|\Sigma|_B} \rtimes |j|$ ,  $H$  is a homotopy from  $(\text{id}_{|\Sigma|_B} \rtimes |j|) \circ f$  to the identity on  $|\Sigma|_B \rtimes |\Delta^n|$  that fixes  $|\Sigma|_B \rtimes |\Lambda_k^n|$ , and

Write  $g_T := \text{pr}_1 g$  and  $g_U := \text{pr}_2 g$ . Define  $\tilde{h} = (\tilde{h}_T, \tilde{h}_U)$  as follows. The map  $\tilde{h}_T$  is the composite

$$\tilde{h}_T : |\Sigma|_B \rtimes |\Delta^n| \xrightarrow{r} |\Sigma|_B \rtimes |\Lambda_k^n| \xrightarrow{g_T} T.$$

The map  $\tilde{h}_U : |\Sigma|_B \rtimes |\Delta^n| \rightarrow \text{Map}([0, 1], U)$  is defined by the sending  $a \in |\Sigma|_B \rtimes |\Delta^n|$  to the path in  $U$  defined by

$$t \mapsto \begin{cases} g_U(r(a))(t(1 + d(a))), & 0 \leq t \leq \frac{1}{1+d(a)} \\ h\left(H\left(a, \frac{1+d(a)}{d(a)}\left(t - \frac{1}{1+d(a)}\right)\right)\right), & \frac{1}{1+d(a)} < t \leq 1. \end{cases}$$

It follows from the definitions that  $\tilde{h}$  makes both the upper and lower triangles in [\(A.2.7\)](#) commute. This completes the proof that  $q_f$  is a  $D_B$ -fibration.  $\square$

Now we use [Lemmas A.1.4](#) and [A.2.6](#) to prove that the projective model structure on  $\text{Fun}(\text{sd}(P)^{\text{op}}, \mathbf{sSet})$  right-transfers to  $\mathbf{Top}_{/|P|}$  and  $\mathbf{Top}_{/P}$ . To apply [Lemma A.1.4](#), we first remark on the presentability of  $\mathbf{Top}$ .

**A.2.8.** Recall that we write  $\mathbf{Top}$  for the category of *numerically generated* topological spaces, which is a presentable category [9; 10, Corollary 3.7]. Consequently the overcategories  $\mathbf{Top}_{/P}$  and  $\mathbf{Top}_{/|P|}$  are presentable for any poset  $P$ .

**A.2.9 Corollary.** *Let  $P$  be a poset. Let  $B$  denote either  $P$  or  $|P|$ . Then the projective model structure on  $\mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{sSet})$  right-transfers along the simplicial adjunction*

$$|-|_B \circ L_P : \mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{sSet}) \rightleftarrows \mathbf{Top}_{/B} : D_B.$$

*Consequently the right-transferred model structure on  $\mathbf{Top}_{/B}$  is combinatorial and simplicial. Moreover, every object of  $\mathbf{Top}_{/B}$  is fibrant, so the right-transferred model structure is right-proper.*

*Proof.* Combine the fact that  $\mathrm{Fun}(\mathrm{sd}(P)^{op}, \mathbf{sSet})^{proj}$  is a combinatorial simplicial model category [HTT, Proposition A.2.8.2 & Remark A.2.8.4] with Lemma A.2.6, Lemma A.1.4, (A.1.2.4), (A.2.3), and (A.1.2.5).  $\square$

**A.2.10. Corollary A.2.9** and (A.1.2) imply that the adjunction  $\mathbf{Top}_{/|P|} \rightleftarrows \mathbf{Top}_{/P}$  is a simplicial Quillen adjunction with respect to the right-transferred model structures.

**A.2.11 Warning.** The right-transferred model structure on  $\mathbf{Top}_{/|P|}$  is importantly *not* the model structure induced on the overcategory  $\mathbf{Top}_{/|P|} = (\mathbf{Top}_{/P})_{/|P|}$  by the right-transferred model structure on  $\mathbf{Top}_{/P}$ . Though these do present the same homotopy theory (as a consequence of Corollary A.3.3), the right-transferred model structure on  $\mathbf{Top}_{/|P|}$  is much more rigid.

### A.3 The Quillen equivalence between $\mathbf{Top}_{/|P|}$ and $\mathbf{Top}_{/P}$

We now prove that the simplicial Quillen adjunction  $\pi_{P,!} : \mathbf{Top}_{/|P|} \rightleftarrows \mathbf{Top}_{/P} : \pi_P^*$  is a Quillen equivalence. This follows easily from the well-known fact that for any poset  $P$ , the map of quasicategories  $\mathrm{Sing}_P |P|_P \rightarrow P$  is a Joyal equivalence (Lemma A.3.2). Since this does not seem to be written down in the literature, for completeness we record Douteau's proof.

**A.3.1 Lemma** ([8, Lemme 7.3.9]). *Let  $P$  be a poset. For any string  $\Sigma \subset P$ , there is a stratified deformation retraction of  $\pi_P^{-1}(\Sigma) \subset |P|_P$  onto  $|\Sigma|_P \subset \pi_P^{-1}(\Sigma)$ .*

*Proof.* We construct a deformation retraction  $H : \pi_P^{-1}(\Sigma) \times [0, 1] \rightarrow \pi_P^{-1}(\Sigma)$  over  $P$  by constructing its restriction to  $|\Sigma|_P \cap \pi_P^{-1}(\Sigma)$  for each string  $S \subset P$  ((1.2.5)). Let

$$S = \{p_0 < \dots < p_m\} \subset P$$

be a string, and write  $I_S := \{i \in [m] \mid p_i \in \Sigma\}$  for the set of indices of elements in  $S \cap \Sigma$ .

Throughout this proof, we regard  $|\Sigma|_P$  as the standard topological  $m$ -simplex of points  $(t_0, \dots, t_m) \in [0, 1]^m$  such that  $\sum_{i=0}^m t_i = 1$ . For each integer  $0 \leq i \leq m$ , define a function  $H_i^S : (|\Sigma|_P \cap \pi_P^{-1}(\Sigma)) \times [0, 1] \rightarrow [0, 1]$  by the formula

$$H_i^S((t_0, \dots, t_m), s) := \begin{cases} (1-s)t_i, & i \in I_S \\ \left(1 + s \frac{\sum_{j \in I_S} t_j}{\sum_{j \notin I_S} t_j}\right) t_i, & i \notin I_S. \end{cases}$$

Now define homotopy

$$H^S : (|S|_P \cap \pi_P^{-1}(\Sigma)) \times [0, 1] \rightarrow |S|_P \cap \pi_P^{-1}(\Sigma) \subset |S|_P$$

by setting

$$H^S((t_0, \dots, t_m), s) := (H_0^S((t_0, \dots, t_m), s), \dots, H_m^S((t_0, \dots, t_m), s)).$$

It is immediate from the definitions that  $H^S$  is a well-defined function to  $|S|_P \cap \pi_P^{-1}(\Sigma)$  and is a homotopy over  $P$ . Note that  $H^S$  provides a homotopy over  $P$  from the identity to a retraction  $|S|_P \cap \pi_P^{-1}(\Sigma) \rightarrow |S|_P \times |\Sigma|_P$ . The homotopies  $H^S$  for  $S \in \text{sd}(P)$  are compatible with inclusions of strings  $S' \subset S$ , hence glue together to define a deformation retraction  $H : \pi_P^{-1}(\Sigma) \times [0, 1] \rightarrow \pi_P^{-1}(\Sigma)$  over  $P$  from  $\pi_P^{-1}(\Sigma)$  onto  $|\Sigma|_P$ .  $\square$

Now we may employ the method of the proof of [HA, Theorem A.6.10].

**A.3.2 Lemma.** *For any poset  $P$ , the map of quasicategories  $\text{Sing}_P |P|_P \rightarrow P$  is a Joyal equivalence, hence a trivial Douteau–Henriques fibration in  $\mathbf{Top}_{/P}$ .*

*Proof.* In light of Lemma A.3.1 and (1.1.9), we are reduced to the case where  $P = [n]$  is a linearly ordered finite poset. Since every stratum of  $|\Delta^n|$  is nonempty, the functor  $\text{Sing}^{[n]} |\Delta^n| \rightarrow [n]$  is essentially surjective. Now we show that it is fully faithful. Let  $i, j \in [n]$  and fix  $x \in |\Delta^n|_i$ , and  $y \in |\Delta^n|_j$ . It is clear that

$$M_{x,y} := \text{Map}_{\text{Sing}^{[n]} |\Delta^n|}(x, y)$$

is empty unless  $i \leq j$ . We wish to prove that  $M_{x,y}$  is contractible if  $i \leq j$ . We can identify  $M_{x,y}$  with  $\text{Sing}(E_{x,y})$ , where  $E_{x,y}$  is the topological space of paths  $\gamma : [0, 1] \rightarrow |\Delta^n|$  such that  $\gamma(0) = x$ ,  $\gamma(1) = y$ , and  $\gamma(t)$  belongs to the stratum  $|\Delta^n|_j$  for all  $t > 0$ . Now observe that there is a contracting homotopy  $h : E_{x,y} \times [0, 1] \rightarrow E_{x,y}$ , given by the formula

$$h(\gamma, s)(t) := (1-s)\gamma(t) + s(1-t)x + st y. \quad \square$$

Lemma A.3.2, (A.1.2.2), and the fact that every object of  $\mathbf{Top}_{/|P|}$  is fibrant now imply:

**A.3.3 Corollary.** *For any poset  $P$  and  $T \in \mathbf{Top}_{/|P|}$ , the unit  $T \rightarrow T \times_P |P|$  is a weak equivalence in  $\mathbf{Top}_{/|P|}$ . Consequently, the simplicial Quillen adjunction*

$$\pi_{P,!} : \mathbf{Top}_{/|P|} \rightleftarrows \mathbf{Top}_{/P} : \pi_P^*$$

*is a Quillen equivalence.*

## A.4 The Quillen equivalence between $\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})$ and $\mathbf{Top}_{/|P|}$

We now prove that the Quillen adjunction  $\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})^{proj} \rightleftarrows \mathbf{Top}_{/|P|}$  is a Quillen equivalence by proving a strong rigidity result about spaces of sections  $|\Sigma| \rightarrow |L_P(F)|$  over  $|P|$ , for  $F$  projective cofibrant (Proposition A.4.7).

**A.4.1.** Since a cofibrant object in the projective model structure on a functor category is necessarily a diagram of cofibrations between cofibrant objects, if  $F$  is a cofibrant object of  $\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})^{proj}$ , then  $F$  is a diagram of monomorphisms.

For the proof of the rigidity result, it will be useful to know that if  $F$  is a diagram of monomorphisms, then the induced morphism  $\Sigma \rtimes F(\Sigma) \rightarrow L_P(F)$  is a monomorphism. This follows from the following elementary lemma (and the fact that colimits in  $\mathbf{sSet}$  are computed pointwise).

**A.4.2 Lemma.** *Let  $I$  be a poset and  $G: I \rightarrow \mathbf{Set}$  a functor. If  $G$  is a diagram of monomorphisms, then for each  $i \in I$ , the natural map  $G(i) \rightarrow \operatorname{colim}_I G$  is a monomorphism.*

**A.4.3 Corollary.** *Let  $P$  be a poset and  $F: \operatorname{sd}(P)^{op} \rightarrow \mathbf{sSet}$  a diagram. If  $F$  is a diagram of monomorphisms, then for every pair of strings  $\Sigma' \subset \Sigma \subset P$ , the induced morphism*

$$\Sigma' \rtimes F(\Sigma) \rightarrow L_P(F)$$

*is a monomorphism in  $\mathbf{sSet}_{/P}$ .*

**A.4.4.** Let  $P$  be a poset and  $\Sigma \subset P$  a string. Note that for any  $X \in \mathbf{Top}_{/|P|}$ , we have an identification

$$\operatorname{Map}_{/|P|}(|\Sigma|, X) = \operatorname{Map}_{/|\Sigma|}(|\Sigma|, |\Sigma| \times_{|P|} X) .$$

**A.4.5.** Let  $P$  be a poset and  $\Sigma \subset P$  a string. Write  $\operatorname{Pair}_\Sigma(P) \subset \operatorname{Pair}(P)$  for the full subposet spanned by those pairs  $(S, S')$  where  $S' \subset \Sigma$ . Note that the inclusion  $\operatorname{Pair}(\Sigma) \subset \operatorname{Pair}_\Sigma(P)$  has a left adjoint given by the assignment  $(S, S') \mapsto (S \cap \Sigma, S')$ . In particular, the inclusion  $\operatorname{Pair}(\Sigma) \subset \operatorname{Pair}_\Sigma(P)$  is colimit-cofinal.

**A.4.6.** Let  $P$  be a poset and  $F: \operatorname{sd}(P)^{op} \rightarrow \mathbf{sSet}$  a functor. It is not difficult to see that for any string  $\Sigma \subset P$ , the space of sections of  $|L_P(F)|$  over  $|\Sigma|$  can be computed as the space of sections of  $|L_\Sigma(F|_{\operatorname{sd}(\Sigma)^{op}})|$ . As a consequence of (A.4.4) we have

$$\operatorname{Map}_{/|P|}(|\Sigma|, |L_P(F)|) \cong \operatorname{Map}_{/|\Sigma|}(|\Sigma|, |\Sigma| \times_{|P|} |L_P(F)|) .$$

Since geometric realization is a left exact functor<sup>6</sup>, we have

$$|\Sigma| \times_{|P|} |L_P(F)| \cong |\Sigma \times_P L_P(F)| .$$

Since  $\mathbf{sSet}$  is a topos, colimits are universal in  $\mathbf{sSet}$ , hence

$$\begin{aligned} \Sigma \times_P L_P(F) &\cong \operatorname{colim}_{(S, S') \in \operatorname{Pair}(P)} (\Sigma \times_P S' \times F(S)) \\ &\cong \operatorname{colim}_{(S, S') \in \operatorname{Pair}(P)} (\Sigma \cap S') \times F(S) \\ &= \operatorname{colim}_{(S, S'') \in \operatorname{Pair}_\Sigma(P)} S'' \times F(S) . \end{aligned}$$

Since the inclusion  $\operatorname{Pair}(\Sigma) \subset \operatorname{Pair}_\Sigma(P)$  is colimit-cofinal (A.4.5), we see that

$$\Sigma \times_P L_P(F) \cong \operatorname{colim}_{(S, S') \in \operatorname{Pair}(\Sigma)} S' \times F(S) = L_\Sigma(F|_{\operatorname{sd}(\Sigma)^{op}}) .$$

Hence

$$\operatorname{Map}_{/|P|}(|\Sigma|, |L_P(F)|) \cong \operatorname{Map}_{/|\Sigma|}(|\Sigma|, |L_\Sigma(F|_{\operatorname{sd}(\Sigma)^{op}})|) .$$

<sup>6</sup>For this we are using the fact that we are working with a ‘convenient category’ of topological spaces (see [11, Theorem 4.3.16; 12, Chapter III, §3]).

**A.4.7 Proposition.** *Let  $P$  be a poset,  $\Sigma \subset P$  a string, and  $F: \text{sd}(P)^{op} \rightarrow \mathbf{sSet}$  a diagram of monomorphisms. Then every map  $|\Sigma| \rightarrow |L_P(F)|$  over  $|P|$  factors through the closed subspace  $|\Sigma| \times |F(\Sigma)| \subset |L_P(F)|$ . Hence the subspace inclusion*

$$\text{Map}_{/|P|}(|\Sigma|, |\Sigma| \times |F(\Sigma)|) \hookrightarrow \text{Map}_{/|P|}(|\Sigma|, |L_P(F)|)$$

*is a homeomorphism. Consequently, we have a natural homeomorphism*

$$(A.4.8) \quad |F(\Sigma)| \simeq \text{Map}_{/|P|}(|\Sigma|, |L_P(F)|).$$

*Proof.* By (A.4.5) it suffices to consider the case that  $P = \Sigma$  is a linearly ordered set of finite cardinality. If  $\#\Sigma = 0$ , then the claim is obvious, so assume that  $\#\Sigma \geq 1$ .

Let  $\sigma: |\Sigma| \rightarrow |L_\Sigma(F)|$  be a section of the structure map  $s: |L_\Sigma(F)| \rightarrow |\Sigma|$ ; we show that  $\sigma$  factors through the closed subspace  $|\Sigma| \times |F(\Sigma)|$  of  $|L_\Sigma(F)|$ . Since  $|\Sigma|$  is compact Hausdorff and  $|L_\Sigma(F)|$  is Hausdorff,  $\sigma$  is closed. Since  $\sigma$  is a monomorphism,  $\sigma$  is a closed embedding. Let  $\text{int}(|\Sigma|)$  denote the interior of  $|\Sigma|$ . Since

$$|L_\Sigma(F)| \cong \text{colim}_{(S, S') \in \text{Pair}(\Sigma)} |S'| \times |F(S)|$$

is the colimit of a diagram of closed embeddings, we see that

$$s^{-1}(\text{int}(|\Sigma|)) = \text{int}(|\Sigma|) \times |F(\Sigma)| \subset |L_\Sigma(F)|.$$

Hence the restriction of  $\sigma$  to  $\text{int}(|\Sigma|)$  factors through  $\text{int}(|\Sigma|) \times |F(\Sigma)|$ . Since  $\sigma$  is a closed embedding,  $\sigma$  factors through the closed subspace  $|\Sigma| \times |F(\Sigma)|$  of  $|L_\Sigma(F)|$ .  $\square$

**A.4.9 Remark.** The argument in the proof of Proposition A.4.7 works without the reduction to the case  $P = \Sigma$ , but we found it much easier to see in this case.

**A.4.10 Theorem** ([8, Théorème 7.3.10]). *For any poset  $P$ , the simplicial Quillen adjunctions*

$$\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})^{proj} \rightleftarrows \mathbf{Top}_{/|P|} \quad \text{and} \quad \text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})^{proj} \rightleftarrows \mathbf{Top}_{/P}$$

*are Quillen equivalences between combinatorial simplicial model categories.*

*Proof.* By Corollary A.3.3 it suffices to prove that the left-hand Quillen adjunction is a Quillen equivalence. Since the Douteau–Henriques model structure on  $\mathbf{Top}_{/|P|}$  is right-transferred from the projective model structure on  $\text{Fun}(\text{sd}(P)^{op}, \mathbf{sSet})$ , it suffices to show that for any projective cofibrant object  $F: \text{sd}(P)^{op} \rightarrow \mathbf{sSet}$  and string  $\Sigma \subset P$ , the morphism

$$\eta_F(\Sigma): F(\Sigma) \rightarrow \text{Sing Map}_{/|P|}(|\Sigma|, |L_P(F)|)$$

induced by the unit of the adjunction is a Kan equivalence (A.1.2.3). In light of natural homeomorphism (A.4.8) provided by (A.4.1) and Proposition A.4.7, we see that the map  $\eta_F(\Sigma)$  is simply given by the Kan equivalence  $F(\Sigma) \rightarrow \text{Sing } |F(\Sigma)|$ .  $\square$

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