Extended étale homotopy groups from profinite Galois categories

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January 20, 2019

Abstract

In this note we show that the protruncated shape of a spectral ∞-topos is a delocalization of its *profinite* stratified shape. This gives a way to reconstruct the extended étale homotopy groups (i.e., the *non*-profinitely complete étale homotopy groups) of a coherent scheme from its *profinite* Galois category.

Contents

Introduction

Let X be a coherent (i.e., quasicompact quasiseparated) scheme. In recent work with Clark Barwick and Saul Glasman [3], we constructed a *delocalization* of the profinite completion of the Artin–Mazur–Friedlander étale homotopy type of X [1; 5]. We call this delocalization the *profinite Galois category* $\mathrm{Gal}(X)$ of X. The profinite Galois category $\mathrm{Gal}(X)$ is pro-object in finite categories, or, equivalently, a category object in profinite topological spaces [2; 3, p. 5 & Construction 13.5]. The underling category of $\mathrm{Gal}(X)$ has objects geometric points of X and morphisms specalizations *in the étale topology* (i.e., is the category of points of the étale topos of X). Concretely, given geometric points $x \to X$ and $y \to X$, a morphism $x \to y$ in $\mathrm{Gal}(X)$ is a lift $y \to X_{(x)}$ of the geometric point $y \to X$ to the strict localization $X_{(x)}$ of X at x. The topology on $\mathrm{Gal}(X)$

globalizes the profinite topology on the absolute Galois group $\mathrm{Gal}(\kappa(x_0)^{sep}/\kappa(x_0))$ of the residue field $\kappa(x_0)$ at each point $x_0 \in X$.

From the profinite category $\mathrm{Gal}(X)$ we can extract a prospace $H(\mathrm{Gal}(X))$ by formally inverting all morphisms. Our delocalization result [3, Examples 11.6 & 13.6] says that $H(\mathrm{Gal}(X))$ and the étale homotopy type of X become (canonically) equivalent after profinite completion. In this note we provide a stronger relationship between the prospace $H(\mathrm{Gal}(X))$ and the étale homotopy type: they agree up to *protruncation*. Morphisms in the ∞-category $\mathrm{Pro}(\mathbf{Spc})$ of prospaces that induce equivalences after protruncation are precisely those morphisms that become $\natural$-*isomorphisms* in the category $\mathrm{Pro}(h\mathbf{Spc})$, in the terminology of Artin–Mazur [1, Definition 4.2].

A Theorem. *Let X be a coherent scheme and write $\Pi_\infty^{ét}(X) \in \mathrm{Pro}(\mathbf{Spc})$ for the étale homotopy type of X. Then there is a natural natural map of prospaces*

$$\theta_X \colon \Pi_\infty^{ét}(X) \to H(\mathrm{Gal}(X)) \,.$$

Moreover, θ_X induces an equivalence on protruncations. As a consequence:

- *For each integer $n \geq 1$ and geometric point $x \to X$, we have canonical isomorphisms of progroups*
$$\pi_n^{ét}(X, x) \simeq \pi_n(H(\mathrm{Gal}(X)), x) \,,$$
where $\pi_n^{ét}(X, x)$ is the n^{th} homotopy progroup of the étale homotopy type of X.

- *For any ring R, there is an equivalence of ∞-categories between local systems of R-modules on X that are uniformly bounded both below and above and continuous functors $\mathrm{Gal}(X) \to D^b(R)$ that carry every morphism to an equivalence.*

The progroups $\pi_n^{ét}(X, x)$ are what we call the *extended étale homotopy groups* of X. Note that the progroup $\pi_1^{ét}(X, x)$ is the *groupe fondamentale élargi* of [SGA 3_{II}, Exposé X, §6]; the usual étale fundamenal group of [SGA 1, Exposé V, §7] is the profinite completion of $\pi_1^{ét}(X, x)$.

While the protruncated étale homotopy type of a connected Noetherian geometrically unibranch scheme is already profinite [1, Theorem 11.1; 5, Theorem 7.3; DAG XIII, Theorem 3.6.5], in general Theorem A provides more refined information about the étale homotopy type, as illustrated in the following example.

B Example. Consider the nodal cubic curve

$$C = \mathrm{Spec}(\mathbf{C}[x, y]/(y^2 - x^2(x + 1))$$

over the complex numbers. The Riemann Existence Theorem [1, Theorem 12.9; 4, Proposition 4.12; 5, Theorem 8.6] implies that the *profinite completion* of the étale homotopy type of C is equivalent to the profinite completion of the circle S^1. It is well-known that, in fact, the *protruncation* of the étale homotopy type of C is S^1; Theorem A provides an easy 'categorical' explanation of this fact.

There is a continuous functor from $\mathrm{Gal}(C)$ to the poset category $\{0 < 1\}$ given by sending the node point to 0 and every other geometric point to 1. The local ring $O_{C,(x,y)}$

at the node point has two prime ideals and the strict Henselization of $O_{C,(x,y)}$ is isomorphic to the strict Henselization of

$$(\mathbf{C}[u,v]/(uv))_{(u,v)} .$$

Using this one sees that there are two lifts of the generic geometric point of C to the strict localization of C at the node. Hence the continuous functor $\mathrm{Gal}(C) \to \{0 < 1\}$ factors through the category D with two objects 0 and 1 and two distinct morphisms $0 \rightrightarrows 1$. Moreover, the functor $\mathrm{Gal}(C) \to D$ induces an equivalence on underlying homotopy types: the prospace $H(\mathrm{Gal}(C))$ is equivalent to $H(D) \simeq S^1$. Theorem A now shows that the protruncation of the étale homotopy type of the nodal cubic is S^1.

We relate the étale homotopy type and profinite Galois category of a coherent scheme by situating the problem in a more general context. In [3] we provided an equivalence of ∞-categories

$$\widetilde{(-)} \colon \mathrm{Pro}(\mathbf{Str}_\pi) \simeq \mathbf{StrTop}_\infty^{spec}$$

between the ∞-*category of profinite stratified spaces* (on the left) and the ∞-*category of spectral stratified* ∞-*topoi* (on the right) [3, Theorem 10.10]. The primary example of a spectral stratified ∞-topos is the étale ∞-topos $X_{\acute{e}t}$ of a coherent scheme X with its natural stratification by the Zariski space of X [3, Example 10.6]. The corresponding profinite stratified space is the profinite Galois category $\mathrm{Gal}(X)$ [3, Construction 13.5].

The equivalence $\mathrm{Pro}(\mathbf{Str}_\pi) \simeq \mathbf{StrTop}_\infty^{spec}$ provides a way to reconstruct the prospace given by the shape of the étale ∞-topos of a coherent scheme X[1] from its profinite Galois category $\mathrm{Gal}(X)$, via the composite

$$\mathrm{Pro}(\mathbf{Str}_\pi) \xrightarrow{\sim} \mathbf{StrTop}_\infty^{spec} \longrightarrow \mathbf{Top}_\infty \xrightarrow{\Pi_\infty} \mathrm{Pro}(\mathbf{Spc}) ,$$

where the middle functor functor forgets the stratification, and Π_∞ is the shape (see Definition 1.3). There's another functor $H \colon \mathrm{Pro}(\mathbf{Str}_\pi) \to \mathrm{Pro}(\mathbf{Spc})$ that doesn't require the use of ∞-topoi, namely, the extension to pro-objects of the composite

$$\mathbf{Str}_\pi \longrightarrow \mathbf{Cat}_\infty \xrightarrow{H} \mathbf{Spc} ,$$

where the first functor forgets the stratification and the second functor sends an ∞-category C to the homotopy type $H(C)$ obtained by inverting every morphism in C. It follows formally that these two functors agree on $\mathbf{Str}_\pi$. Moreover, as the extension to pro-objects of a functor $\mathbf{Str}_\pi \to \mathbf{Spc}$, the functor $H \colon \mathrm{Pro}(\mathbf{Str}_\pi) \to \mathrm{Pro}(\mathbf{Spc})$ preserves inverse limits. Thus we have a map

$$\theta_C \colon \Pi_\infty(\widetilde{C}) \to H(C)$$

natural in $C \in \mathrm{Pro}(\mathbf{Str}_\pi)$. In this note we prove that this map is an equivalence after protruncation:

[1] This is, up to protruncation, the Artin–Mazur–Friedlander étale homotopy type of X; see [6, §5], which we recall in Examples 1.6 and 1.9.

C Theorem (Theorem 2.5). *Let* $\mathbf{Spc}_{<\infty} \subset \mathbf{Spc}$ *denote the* ∞*-category of truncated spaces, and write* $\tau_{<\infty}\colon \mathrm{Pro}(\mathbf{Spc}) \to \mathrm{Pro}(\mathbf{Spc}_{<\infty})$ *for the left adjoint to the inclusion. For any profinite stratified space* C*, the natural map*

$$\tau_{<\infty}\theta_C\colon \tau_{<\infty}\Pi_\infty(\widetilde{C}) \to \tau_{<\infty}H(C)$$

of protruncated spaces is an equivalence.

In light of [3, Construction 13.5], Theorem A is immediate from Theorem C.

Since the functor H and the shape Π_∞ agree on $\mathbf{Str}_\pi$ and both H and $\tau_{<\infty}$ preserve inverse limits, by the universal property of the ∞-category of pro-objects, Theorem C follows once we know that the the protruncated shape $\tau_{<\infty}\Pi_\infty$ preserves inverse limits. The forgetful functor $\mathbf{StrTop}_\infty^{spec} \to \mathbf{Top}_\infty$ factors through the subcategory $\mathbf{Top}_\infty^{bc} \subset \mathbf{Top}_\infty$ of bounded coherent ∞-topoi and coherent geometric morphisms. Theorem C thus reduces to the following fact.

D Theorem (Proposition 2.2). *The protruncated shape*

$$\tau_{<\infty}\Pi_\infty\colon \mathbf{Top}_\infty^{bc} \to \mathrm{Pro}(\mathbf{Spc}_{<\infty})$$

preserves inverse limits.

In § 1 we review the necessary background on pro-objects and shape theory. The familiar reader should skip straight to §2 where we prove Theorems C and D.

Acknowledgments. We thank Clark Barwick for his guidance and sharing his many insights about this material. We also gratefully acknowledge support from both the MIT Dean of Science Fellowship and NSF Graduate Research Fellowship.

1 Preliminaries on shapes & protruncated spaces

In this section we review ∞-categories of pro-objects and shape theory for ∞-topoi. We then record some facts about protruncations that we'll need.

Review of shape theory

1.1. We say that a small ∞-category I is ***inverse*** if the opposite ∞-category I^{op} is filtered. An ***inverse system*** in an ∞-category C is a functor $I \to C$, where I is an inverse ∞-category. An ***inverse limit*** is a limit of an inverse system.

Let C be an ∞-category. We write $\mathrm{Pro}(C)$ for the ∞-category of ***pro-objects*** in C obtained by freely adjoining inverse limits to C, and $j\colon C \to \mathrm{Pro}(C)$ for the Yoneda embedding. We say that a pro-object $X \in \mathrm{Pro}(C)$ is ***constant*** if X lies in the essential image of $j\colon C \to \mathrm{Pro}(C)$. If $X\colon I \to C$ is an inverse system, we write $\{X_i\}_{i\in I} \coloneqq \lim_{i\in I} j(X_i)$ for the pro-object it defines.

If C is accessible and admits finite limits, then $\mathrm{Pro}(C)$ is equivalent to the full subcategory of $\mathrm{Fun}(C, \mathbf{Spc})^{op}$ spanned by the left exact accessible functors [SAG, Proposition A.8.1.6]. Let $f\colon C \to D$ be a left exact accessible functor between accessible ∞-categories which admit small limits. Then the functor $f\colon \mathrm{Pro}(C) \to \mathrm{Pro}(D)$ admits a left adjoint $L\colon \mathrm{Pro}(D) \to \mathrm{Pro}(C)$ [SAG, Example A.8.1.8]. We refer to $L \circ j\colon D \to \mathrm{Pro}(C)$ as the ***pro-left adjoint*** of f.

1.2 Notation. We write $\mathbf{Cat}_\infty$ for the ∞-category of ∞-categories and $\mathbf{Spc} \subset \mathbf{Cat}_\infty$ for the full subcategory spanned by the ∞-groupoids, i.e., the ∞-category of spaces.

We write $\mathbf{Top}_\infty \subset \mathbf{Cat}_\infty$ for the ∞-category of ∞-topoi and geometric morphisms. For any ∞-topos $\boldsymbol{X}$, we write $\Gamma_{\boldsymbol{X},*}$ or Γ_* for the global sections geometric morphism, which is the essentially unique geometric morphism $\boldsymbol{X} \to \mathbf{Spc}$.

1.3 Definition. The ***shape*** $\Pi_\infty\colon \mathbf{Top}_\infty \to \mathrm{Pro}(\mathbf{Spc})$ is the left adjoint to the extension to pro-objects of the fully faithful functor $\mathbf{Spc} \hookrightarrow \mathbf{Top}_\infty$ given by $K \mapsto \mathrm{Fun}(K, \mathbf{Spc})$ [SAG, §E.2.2]. The shape admits two other very useful descriptions:

- Let $\boldsymbol{X}$ be an ∞-topos, and write $\Gamma_!\colon \boldsymbol{X} \to \mathrm{Pro}(\mathbf{Spc})$ for the pro-left adjoint of $\Gamma^*\colon \mathbf{Spc} \to \boldsymbol{X}$. The shape of $\boldsymbol{X}$ is equivalent to the prospace $\Gamma_!(1)$, where $1 \in \boldsymbol{X}$ denotes the terminal object [HA, Remark A.1.10; 6, §2].

- As a left exact accessible functor $\mathbf{Spc} \to \mathbf{Spc}$, the prospace $\Pi_\infty(\boldsymbol{X})$ is the composite $\Gamma_*\Gamma^*$ [HTT, §7.1.6; 6, §2].

1.4 Notation. We write $H\colon \mathbf{Cat}_\infty \to \mathbf{Spc}$ for the left adjoint to the inclusion. The ∞-groupoid $H(C)$ is given by the colimit $H(C) \simeq \mathrm{colim}_C 1_{\mathbf{Spc}}$ of the constant diagram $C \to \mathbf{Spc}$ at the terminal object $1_{\mathbf{Spc}} \in \mathbf{Spc}$.

1.5 Example. If C is a small ∞-category, then $\Gamma^*\colon \mathbf{Spc} \to \mathrm{Fun}(C, \mathbf{Spc})$ admits a genuine left adjoint $\Gamma_!\colon \mathrm{Fun}(C, \mathbf{Spc}) \to \mathbf{Spc}$ given by taking the colimit of a diagram $C \to \mathbf{Spc}$. The shape of the ∞-topos $\mathrm{Fun}(C, \mathbf{Spc})$ is thus given by the colimit of the constant diagram at the terminal object of $\mathbf{Spc}$:

$$\Pi_\infty(\mathrm{Fun}(C, \mathbf{Spc})) = \Gamma_!(1_{\mathrm{Fun}(C,\mathbf{Spc})}) = \mathrm{colim}_C 1_{\mathbf{Spc}} \simeq H(C)\ .$$

Moreover, the functor $H\colon \mathbf{Cat}_\infty \to \mathbf{Spc}$ is equivalent to the composite

$$\mathbf{Cat}_\infty \xrightarrow{\mathrm{Fun}(-,\mathbf{Spc})} \mathbf{Top}_\infty \xrightarrow{\Pi_\infty} \mathbf{Spc}\ .$$

1.6 Example ([6, Corollary 5.6])**.** If X is a locally Noetherian scheme, then the Artin–Mazur–Friedlander étale homotopy type of X corepresents the shape of the *hypercomplete*[2] étale ∞-topos $X_{\acute{e}t}^{hyp}$ of X.

The shape of the étale ∞-topos $X_{\acute{e}t}$ of X agrees with the Artin–Mazur-Friedlander étale homotopy type up to *protruncation* (Example 1.9), to which we now turn.

Protruncated objects

In this subsection, we recall some facts about protruncated objects and record an interesting observation (Lemma 1.11) that we couldn't locate in the literature.

[2] See [HTT, §6.5.2] for a treatment of hypercomplete ∞-topoi.

1.7 Notation. Let C be a presentable ∞-category. For each integer $n \geq -2$, write $C_{\leq n} \subset C$ for the full subcategory spanned by the n-truncated objects, and $\tau_{\leq n} \colon C \to C_{\leq n}$ for the n-truncation functor, which is left adjoint to the inclusion $C_{\leq n} \subset C$ [HTT, Proposition 5.5.6.18]. Write $C_{<\infty} \subset C$ for the full subcategory spanned by those objects which are n-truncated for some integer $n \geq -2$.

The ***pro-n-truncation*** functor $\tau_{\leq n} \colon \mathrm{Pro}(C) \to \mathrm{Pro}(C_{\leq n})$ is the extension of the n-truncation functor $\tau_{\leq n} \colon C \to C_{\leq n}$ to pro-objects.

1.8. Let C be a presentable ∞-category. Then the extension to pro-objects of the functor $C \to \mathrm{Pro}(C_{<\infty})$ given by sending an object $X \in C$ to the inverse system given by its Postnikov tower $\{\tau_{\leq n}(X)\}_{n \geq -2}$ is left adjoint to the inclusion $\mathrm{Pro}(C_{<\infty}) \hookrightarrow \mathrm{Pro}(C)$. We call this left adjoint $\tau_{<\infty} \colon \mathrm{Pro}(C) \to \mathrm{Pro}(C_{<\infty})$ ***protruncation***.

A morphism of pro-objects $f \colon X \to Y$, regarded as left exact accessible functors $C \to \mathbf{Spc}$, is an equivalence after protuncation if and only if for every truncated object $K \in C_{<\infty}$, the induced morphism $f(K) \colon X(K) \to Y(K)$ is an equivalence.

1.9 Example. Since truncated objects are hypercomplete, for any ∞-topos $\boldsymbol{X}$, the inclusion $\boldsymbol{X}^{hyp} \hookrightarrow \boldsymbol{X}$ of the ∞-topos of hypercomplete objects of $\boldsymbol{X}$ induces an equivalence

$$\tau_{<\infty}\Pi_\infty(\boldsymbol{X}^{hyp}) \simeq \tau_{<\infty}\Pi_\infty(\boldsymbol{X})$$

on protruncated shapes. In light of Example 1.6, the shape of the étale ∞-topos of a locally Noetherian scheme X agrees with the Artin–Mazur–Friedlander étale homotopy type of X after protruncation.

For an arbitrary scheme X, we simply refer to the shape $\Pi_\infty(X_{ét})$ of the étale ∞-topos $X_{ét}$ of X as the ***étale homotopy type*** of X.

1.10. Let C be a presentable ∞-category. The essentially unique functor $\mathrm{Pro}(C) \to C$ that perserves inverse limits and restricts to the identity $C \to C$ is right adjoint to the Yoneda embedding $j \colon C \hookrightarrow \mathrm{Pro}(C)$ [SAG, Example A.8.1.7]. Hence we have adjunctions

$$C \underset{\longleftarrow}{\overset{j}{\longrightarrow}} \mathrm{Pro}(C) \underset{\longleftarrow}{\overset{\tau_{<\infty}}{\longrightarrow}} \mathrm{Pro}(C_{<\infty})\ .$$

If Postnikov towers converge in C, i.e., C is a Postnikov complete presentable ∞-category [SAG, Definition A.7.2.1], then the composite left adjoint is also fully faithful:

1.11 Lemma. *Let C be a Postnikov complete presentable ∞-category (e.g., a Postnikov complete ∞-topos). Then the protruncation functor*

$$\tau_{<\infty} \colon C \to \mathrm{Pro}(C_{<\infty})$$

is fully faithful. Moreover, the essential image of $\tau_{<\infty} \colon C \hookrightarrow \mathrm{Pro}(C_{<\infty})$ is the full subcategory spanned by those protruncated objects X such that for each integer $n \geq -2$, the pro-n-truncation $\tau_{\leq n}(X) \in \mathrm{Pro}(C_{\leq n})$ is a constant pro-object.

1.12. Composing the fully faithful functor $\tau_{<\infty} \colon \mathbf{Spc} \hookrightarrow \mathrm{Pro}(\mathbf{Spc}_{<\infty})$ with the inclusion $\mathrm{Pro}(\mathbf{Spc}_{<\infty}) \hookrightarrow \mathrm{Pro}(\mathbf{Spc})$ gives another embedding of spaces into prospaces: for a space K, the natural morphism of prospaces $j(K) \to \tau_{<\infty}(K)$ is an equivalence if and only if K is truncated. Unlike the Yoneda embedding, the functor $\tau_{<\infty} \colon \mathbf{Spc} \hookrightarrow \mathrm{Pro}(\mathbf{Spc})$ is neither a left nor a right adjoint.

2 Limits & the protruncated shape

The shape does not preseve inverse limits, even of bounded coherent ∞-topoi. In this section we prove that, nevertheless, the *protruncated* shape preserves inverse limits of bounded coherent ∞-topoi. Our main theorem (Theorem 2.5) is an easy consequence.

2.1 Notation. Write $\mathbf{Top}_\infty^{bc} \subset \mathbf{Top}_\infty$ for the subcategory of *bounded coherent* ∞-topoi and *coherent* geometric morphisms [SAG, Definitions A.2.0.12 & A.7.1.2; 3, Definition 5.28].

2.2 Proposition. *The protruncated shape*

$$\tau_{<\infty}\Pi_\infty \colon \mathbf{Top}_\infty^{bc} \to \mathrm{Pro}(\mathbf{Spc}_{<\infty})$$

preserves inverse limits.

Proof. Let $\boldsymbol{X} \colon I \to \mathbf{Top}_\infty^{bc}$ be an inverse system of bounded coherent ∞-topoi and coherent geometric morphisms. For each $i \in I$, the forgetful functor $I_{/i} \to I$ is limit-cofinal [HTT, Example 5.4.5.9 & Lemma 5.4.5.12], so we may without loss of generality assume that I admits a terminal object 1. For each $i \in I$, write

$$\pi_{i,*} \colon \lim_{j\in I} \boldsymbol{X}_j \to \boldsymbol{X}_i$$

for the projection, $\Gamma_{i,*} \coloneqq \Gamma_{\boldsymbol{X}_i,*}$, and $f_{i,*} \colon \boldsymbol{X}_i \to \boldsymbol{X}_1$ for the geometric morphism induced by the essentially unique morphism $i \to 1$ in I. Write $\Gamma_* \colon \lim_{j\in I} \boldsymbol{X}_j \to \mathbf{Spc}$ for the global sections geometric morphism.

We want to show that the natural morphism

$$\operatorname*{colim}_{i\in I^{op}} \Gamma_{i,*}\Gamma_i^* \to \Gamma_*\Gamma^*$$

in $\mathrm{Fun}(\mathbf{Spc}, \mathbf{Spc})$ is an equivalence when restricted to truncated spaces (1.8). By [3, Lemma 8.11] the natural morphism

$$\operatorname*{colim}_{i\in I^{op}} f_{i,*}f_i^* \to \pi_{1,*}\pi_1^*$$

is an equivalence in $\mathrm{Fun}(\boldsymbol{X}_1, \boldsymbol{X}_1)$. Since $\boldsymbol{X}_1$ is bounded coherent, the global sections functor $\Gamma_{1,*} \colon \boldsymbol{X}_1 \to \mathbf{Spc}$ preserves filtered colimits of uniformly truncated objects [SAG, Proposition A.2.3.1; 3, Corollary 5.55]. Thus for any truncated space K we see that

$$\begin{aligned}
\operatorname*{colim}_{i\in I^{op}} \Gamma_{i,*}\Gamma_i^*(K) &\simeq \operatorname*{colim}_{i\in I^{op}} \Gamma_{1,*}f_{i,*}f_i^*\Gamma_1^*(K) \\
&\simeq \Gamma_{1,*}\left(\operatorname*{colim}_{i\in I^{op}} f_{i,*}f_i^*\Gamma_1^*(K)\right) \\
&\simeq \Gamma_{1,*}\circ\left(\operatorname*{colim}_{i\in I^{op}} f_{i,*}f_i^*\right)\circ\Gamma_1^*(K) \\
&\simeq \Gamma_{1,*}\circ\pi_{1,*}\pi_1^*\circ\Gamma_1^*(K) \\
&\simeq \Gamma_*\Gamma^*(K)\,.
\end{aligned}$$

□

Proof of the Main Theorem

We now prove the main result of this note. Recall that we write

$$\widetilde{(-)} \colon \operatorname{Pro}(\mathbf{Str}_\pi) \xrightarrow{\sim} \mathbf{StrTop}_\infty^{spec}$$

for the equivalence of ∞-categories of [3, Theorem 10.10].

2.3 Lemma. *The square*

$$\begin{array}{ccc} \mathbf{Str}_\pi & \xhookrightarrow{\widetilde{(-)}} & \mathbf{StrTop}_\infty^{spec} \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \Pi_\infty} \\ \mathbf{Spc} & \xhookrightarrow[j]{} & \operatorname{Pro}(\mathbf{Spc}) \end{array}$$

commutes.

Proof. By the definition of the equivalence $\operatorname{Pro}(\mathbf{Str}_\pi) \xrightarrow{\sim} \mathbf{StrTop}_\infty^{spec}$ of [3, Theorem 10.10], the following square commutes

$$\begin{array}{ccc} \mathbf{Str}_\pi & \xhookrightarrow{\widetilde{(-)}} & \mathbf{StrTop}_\infty^{spec} \\ \downarrow & & \downarrow \\ \mathbf{Cat}_\infty & \xrightarrow[\operatorname{Fun}(-,\mathbf{Spc})]{} & \mathbf{Top}_\infty \end{array} ,$$

where the vertical functors forget stratifications. Combining this with Example 1.5 proves the claim. □

2.4. Since the extension of $H \colon \mathbf{Str}_\pi \to \mathbf{Spc}$ to pro-objects preserves inverse limits, Lemma 2.3 shows that we have a morphism of prospaces

$$\theta_C \colon \Pi_\infty(\widetilde{C}) \to H(C)$$

natural in $C \in \operatorname{Pro}(\mathbf{Str}_\pi)$.

2.5 Theorem. *For any profinite stratified space C, the natural map*

$$\tau_{<\infty}\theta_C \colon \tau_{<\infty}\Pi_\infty(\widetilde{C}) \to \tau_{<\infty}H(C)$$

of protruncated spaces is an equivalence.

Proof. Since the forgetful functor $\mathbf{StrTop}_\infty^{spec} \to \mathbf{Top}_\infty^{bc}$ preserves inverse limits, Proposition 2.2 implies that the protruncated shape $\tau_{<\infty}\Pi_\infty \colon \mathbf{StrTop}_\infty^{spec} \to \operatorname{Pro}(\mathbf{Spc}_{<\infty})$ preserves inverse limits. Both $\tau_{<\infty}$ and H preserve inverse limits, hence their composite $\tau_{<\infty}H \colon \operatorname{Pro}(\mathbf{Str}_\pi) \to \operatorname{Pro}(\mathbf{Spc}_{<\infty})$ preserves inverse limits. The claim now follows from the fact that θ_C is an equivalence for $C \in \mathbf{Str}_\pi$ (Lemma 2.3) and the universal property of the ∞-category $\operatorname{Pro}(\mathbf{Str}_\pi)$ of profinite stratified spaces. □

2.6. Note that Theorem A from the introduction is immediate from Theorem 2.5, [3, Construction 13.5], and the definition of the étale homotopy type in terms of shape theory (Examples 1.6 and 1.9).

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